The Hardy inequality and the heat equation in twisted tubes

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Abstract

We show that a twist of a three-dimensional tube of uniform cross-section yields an improved decay rate for the heat semigroup associated with the Dirichlet Laplacian in the tube. The proof employs Hardy inequalities for the Dirichlet Laplacian in twisted tubes and the method of self-similar variables and weighted Sobolev spaces for the heat equation.

Contents

1 Introduction

It has been shown recently in [7] that a local twist of a straight three-dimensional tube $\Omega_0 := \mathbb{R} \times \omega$ of non-circular cross-section $\omega \subset \mathbb{R}^2$ leads to an effective repulsive interaction in the Schrödinger equation of a quantum particle constrained to the twisted tube Ω_θ. More precisely, there is a Hardy-type inequality for the particle Hamiltonian modelled by the Dirichlet Laplacian $-\Delta_D^{\Omega_\theta}$ at its threshold energy E_1 if, and only if, the tube is twisted (*cf* Figure 1). That is, the inequality

$$-\Delta_D^{\Omega_\theta} - E_1 \geq \varrho \tag{1.1}$$

holds true, in the sense of quadratic forms in $L^2(\Omega_\theta)$, with a positive function ϱ provided that the tube is twisted, while ϱ is necessarily zero for Ω_0. Here E_1 coincides with the first eigenvalue of the Dirichlet Laplacian $-\Delta_D^{\omega}$ in the cross-section ω.

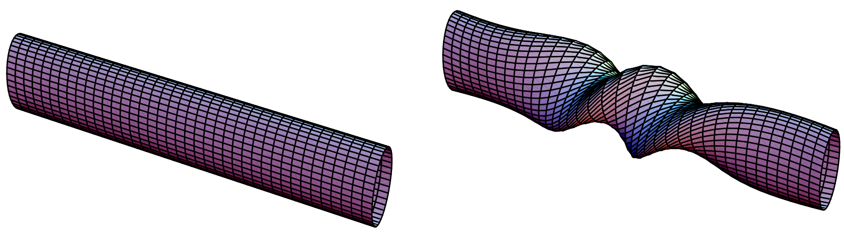

Figure 1: Untwisted and twisted tubes of elliptical cross-section.

The inequality (1.1) has important consequences for conductance properties of quantum waveguides. It clearly implies the absence of bound states (*i.e.*, stationary solutions to the Schrödinger equation) below the energy E_1 even if the particle is subjected to a small attractive interaction, which can be either of potential or geometric origin (*cf* [7] for more details). At the same time, a repulsive effect of twisting on eigenvalues embedded in the essential spectrum has been demonstrated in [14]. Hence, roughly speaking, the twist prevents the particle to be trapped in the waveguide. Additional spectral properties of twisted tubes have been studied in [9, 18, 2].

It is natural to ask whether the repulsive effect of twisting demonstrated in [7] in the quantum context has its counterpart in other areas of physics, too. The present paper gives an affirmative answer to this question for systems modelled by the diffusion equation in the tube Ω_θ:

$$u_t - \Delta u = 0 \,, \tag{1.2}$$

subject to Dirichlet boundary conditions on $\partial\Omega_\theta$. Indeed, we show that the twist is responsible for a faster convergence of the solutions of (1.2) to the (zero) stable equilibrium. The second objective of the paper is to give a new (simpler and more direct) proof of the Hardy inequality (1.1) under weaker conditions than those in [7].

1.1 The main result

Before stating the main result about the large time behaviour of the solutions to (1.2), let us make some comments on the subtleties arising with the study of the heat equation in Ω_θ.

The specific deformation Ω_θ of Ω_0 via twisting we consider can be visualized as follows: instead of simply translating ω along $\mathbb{R}$ we also allow the (non-circular) cross-section ω to rotate with respect to a (non-constant) angle $x_1 \mapsto \theta(x_1)$. See Figure 1 (the precise definition is postponed until Section 2, *cf* Definition 2.1). We assume that the deformation is local, *i.e.*,

$$\dot\theta \text{ has compact support in } \mathbb{R}. \tag{1.3}$$

Then the straight and twisted tubes have the same spectrum (*cf* [17, Sec. 4]):

$$\sigma(-\Delta_D^{\Omega_\theta}) = \sigma_{\mathrm{ess}}(-\Delta_D^{\Omega_\theta}) = [E_1, \infty)\,. \tag{1.4}$$

The fine difference between twisted and untwisted tubes in the spectral setting is reflected in the existence of (1.1) for the former.

In view of the spectral mapping theorem, the indifference (1.4) transfers to the following identity for the heat semigroup:

$$\forall t \geq 0\,, \qquad \left\| e^{\Delta_D^{\Omega_\theta} t} \right\|_{L^2(\Omega_\theta) \to L^2(\Omega_\theta)} = e^{-E_1 t}\,, \tag{1.5}$$

irrespectively whether the tube Ω_θ is twisted or not. That is, we clearly have the exponential decay

$$\|u(t)\|_{L^2(\Omega_\theta)} \leq e^{-E_1 t}\, \|u_0\|_{L^2(\Omega_\theta)} \tag{1.6}$$

for each time $t \geq 0$ and any initial datum u_0 of (1.2). To obtain some finer differences as regards the time-decay of solutions, it is therefore natural to consider rather the *"shifted"* semigroup

$$S(t) := e^{(\Delta_D^\Omega + E_1)t} \tag{1.7}$$

as an operator from a *subspace* of $L^2(\Omega_\theta)$ to $L^2(\Omega_\theta)$.

In this paper we mainly (but not exclusively) consider the subspace of initial data given by the weighted space

$$L^2(\Omega_\theta, K) \qquad \text{with} \qquad K(x) := e^{x_1^2/4}\,, \tag{1.8}$$

and study the asymptotic properties of the semigroup via the *decay rate* defined by

$$\Gamma(\Omega_\theta) := \sup\left\{\Gamma \,\middle|\, \exists C_\Gamma > 0,\, \forall t \geq 0,\; \|S(t)\|_{L^2(\Omega_\theta,K)\to L^2(\Omega_\theta)} \leq C_\Gamma\,(1+t)^{-\Gamma}\right\}.$$

Our main result reads as follows:

Theorem 1.1. *Let $\theta \in C^1(\mathbb{R})$ satisfy (1.3). We have*

$$\Gamma(\Omega_\theta) \begin{cases} = 1/4 & \text{if } \Omega_\theta \text{ is untwisted,} \\ \geq 3/4 & \text{if } \Omega_\theta \text{ is twisted.} \end{cases}$$

The statement of the theorem for solutions u of (1.2) in Ω_θ can be reformulated as follows. For every $\Gamma < \Gamma(\Omega_\theta)$, there exists a positive constant C_Γ such that

$$\|u(t)\|_{L^2(\Omega_\theta)} \leq C_\Gamma \, (1+t)^{-\Gamma} \, e^{-E_1 t} \, \|u_0\|_{L^2(\Omega_\theta,K)} \tag{1.9}$$

for each time $t \geq 0$ and any initial datum $u_0 \in L^2(\Omega_\theta, K)$. This should be compared with the inequality (1.6) which is sharp in the sense that it does not allow for any extra polynomial-type decay rate due to (1.5). On the other hand, we see that the decay rate is at least three times better in a twisted tube provided that the initial data are restricted to the weighted space.

A type of the estimate (1.9) in an untwisted tube can be obtained in a less restrictive weighted space (*cf* Theorem 4.1). The power 1/4 actually reflects the quasi-one-dimensional nature of our model. Indeed, in the whole Euclidean space one has the well known dimensional bound

$$\forall t \geq 0\,, \qquad \left\| e^{\Delta_D^{\mathbb{R}^d} t} \right\|_{L^2(\mathbb{R}^d,K) \to L^2(\mathbb{R}^d)} \, \leq \, (1+t)^{-d/4} \,. \tag{1.10}$$

The fact that the power 1/4 is optimal for untwisted tubes can be established quite easily by a "separation of variables" (*cf* Proposition 4.2). The fine effect of twisting is then reflected in the positivity of $\Gamma(\Omega_\theta) - 1/4$; in view of (1.10), it can be interpreted as "enlarging the dimension" of the tube.

1.2 The idea of the proof

The principal idea behind the main result of Theorem 1.1, *i.e.* the better decay rate in twisted tubes, is the positivity of the function ϱ in (1.1). In fact, Hardy inequalities have already been used as an essential tool to study the asymptotic behaviour of the heat equation in other situations [3, 22]. However, it should be stressed that Theorem 1.1 does not follow as a direct consequence of (1.1) by some energy estimates (*cf* Section 4.3) but that important and further technical developments that we explain now are needed. Nevertheless, overall, the main result of the paper confirms that the Hardy inequalities end up enhancing the decay rate of solutions..

Let us now briefly describe our proof (as given in Section 5) that there is the extra decay rate if the tube is twisted.

I. First, we map the twisted tube Ω_θ to the straight one Ω_0 by a change of variables, and consider rather the transformed (and shifted by E_1) equation

$$u_t - (\partial_1 - \dot\theta \, \partial_\tau)^2 u - \Delta' u - E_1 u = 0 \tag{1.11}$$

in Ω_0 instead of (1.2). Here $-\Delta' := -\partial_2^2 - \partial_3^2$ and $\partial_\tau := x_3\partial_2 - x_2\partial_3$, with $x = (x_1, x_2, x_3) \in \Omega_0$, denote the "transverse" Laplace and angular-derivative operators, respectively.

II. The main ingredient in the subsequent analysis is the method of self-similar solutions developed in the whole Euclidean space by Escobedo and Kavian [8]. Writing

$$\tilde u(y_1, y_2, y_3, s) = e^{s/4} u(e^{s/2} y_1, y_2, y_3, e^s - 1)\,, \tag{1.12}$$

the equation (1.11) is transformed to

$$\tilde u_s - \tfrac{1}{2}\, y_1 \, \partial_1 \tilde u - (\partial_1 - \sigma_s \, \partial_\tau)^2 \tilde u - e^s \, \Delta' \tilde u - E_1 \, e^s \, \tilde u - \tfrac{1}{4} \, \tilde u = 0 \tag{1.13}$$

in self-similarity variables $(y, s) \in \Omega_0 \times (0, \infty)$, where

$$\sigma_s(y_1) := e^{s/2} \dot{\theta}(e^{s/2} y_1) \,. \tag{1.14}$$

Note that (1.13) is a parabolic equation with *time-dependent* coefficients. This non-autonomous feature is a consequence of the non-trivial geometry we deal with and represents thus the main difficulty in our study. We note that an analogous difficulty has been encountered previously for a convection-diffusion equation in the whole space but with a variable diffusion coefficient [5].

III. We reconsider (1.13) in the weighted space (1.8) and show that the associated generator has purely discrete spectrum then. Now a difference with respect to the self-similarity transformation in the whole Euclidean space is that the generator is not a symmetric operator if the tube is twisted. However, this is not a significant obstacle since only the real part of the corresponding quadratic form is relevant for subsequent energy estimates (*cf* (5.11)).

IV. Finally, we look at the asymptotic behaviour of (1.13) as the self-similar time s tends to infinity. Assume that the tube is twisted. The scaling coming from the self-similarity transformation is such that the function (1.14) converges in a distributional sense to a multiple of the delta function supported at zero as $s \to \infty$. The square of σ_s becomes therefore extremely singular at the section $\{0\} \times \omega$ of the tube for large times. At the same time, the prefactors e^s in (1.13) diverge exactly as if the cross-section of the tube shrunk to zero $s \to \infty$. Taking these two simultaneous limits into account, it is expectable that (1.13) will be approximated for large times by the essentially one-dimensional problem

$$\varphi_s - \tfrac{1}{2} \, y_1 \, \varphi_{y_1} - \varphi_{y_1 y_1} - \tfrac{1}{4} \, \varphi = 0 \,, \qquad s \in (0, \infty), \; y_1 \in \mathbb{R} \,, \tag{1.15}$$

with an extra Dirichlet boundary condition at $y_1 = 0$. This evolution equation is explicitly solvable in $L^2(\mathbb{R}, K)$ and it is easy to see that

$$\|\varphi\|_{L^2(\mathbb{R},K)} \leq e^{-\frac{3}{4}s} \, \|\varphi_0\|_{L^2(\mathbb{R},K)} \,, \tag{1.16}$$

for any initial datum φ_0. Here the exponential decay rate transfers to a polynomial one after returning to the original time t, and the number $3/4$ gives rise to that of the bound of Theorem 1.1 in the twisted case.

On the other hand, we get just $1/4$ in (1.16) provided that the tube is untwisted (which corresponds to imposing no extra condition at $y_1 = 0$).

Two comments are in order. First, we do not establish any theorem that solutions of (1.13) can be approximated by those of (1.15) as $s \to \infty$. We only show a strong-resolvent convergence for operators related to their generators (Proposition 5.4). This is, however, sufficient to prove Theorem 1.1 with help of energy estimates. Proposition 5.4 is probably the most significant auxiliary result of the paper and we believe it is interesting in its own right.

Second, in the proof of Proposition 5.4 we essentially use the existence of the Hardy inequality (1.1) in twisted tubes. In fact, the positivity of ϱ is directly responsible for the extra Dirichlet boundary condition of (1.15). Since the Hardy inequality holds in the Hilbert space $L^2(\Omega_0)$ (no weight), Proposition 5.4 is stated for operators transformed to it from (1.8) by an obvious unitary transform. In particular, the asymptotic operator h_D of Proposition 5.4 acts in a different space, $L^2(\mathbb{R})$, but it is unitarily equivalent to the generator of (1.15).

1.3 The content of the paper

The organization of this paper is as follows.

In the following Section 2 we give a precise definition of twisted tubes Ω_θ and the corresponding Dirichlet Laplacian $-\Delta_D^{\Omega_\theta}$.

Section 3 is mainly devoted to a new proof of the Hardy inequality (Theorem 3.1) as announced in [18]. We mention its consequences on the stability of the spectrum of the Laplacian (Proposition 3.2) and emphasize that the Hardy weight cannot be made arbitrarily large by increasing the twisting (Proposition 3.3). Finally, we establish there a new Sobolev-type inequality in twisted tubes (Theorem 3.2).

The heat equation in twisted tubes is considered in Section 4. Using some energy-type estimates, we prove in Theorems 4.1 and 4.2 polynomial-type decay results for the heat semigroup as a consequence of the Sobolev and Hardy inequalities, respectively. Unfortunately, Theorem 4.2 does not represent any improvement upon the 1/4-decay rate of Theorem 4.1 which is valid in untwisted tubes as well.

The main body of the paper is therefore represented by Section 5 where we develop the method of self-similar solutions to get the improved decay rate of Theorem 1.1 as described above. Furthermore, in Section 5.9 we establish an alternative version of Theorem 1.1.

The paper is concluded in Section 6 by referring to physical interpretations of the result and to some open problems.

2 Preliminaries

In this section we introduce some basic definitions and notations we shall use throughout the paper.

2.1 The geometry of a twisted tube

Given a bounded open connected set $\omega \subset \mathbb{R}^2$, let $\Omega_0 := \mathbb{R} \times \omega$ be a straight tube of cross-section ω. We assume no regularity hypotheses about ω. Let $\theta : \mathbb{R} \to \mathbb{R}$ be a C^1-smooth function with bounded derivative (occasionally we will denote by the same symbol θ the function $\theta \otimes 1$ on Ω_0). We introduce another tube of the same cross-section ω as the image

$$\Omega_\theta := \mathcal{L}_\theta(\Omega_0)\,,$$

where the mapping $\mathcal{L}_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\mathcal{L}_\theta(x) := \big(x_1, x_2 \cos\theta(x_1) + x_3 \sin\theta(x_1), -x_2 \sin\theta(x_1) + x_3 \cos\theta(x_1)\big)\,. \quad (2.1)$$

Definition 2.1 (Twisted and untwisted tubes)**.** We say that the tube Ω_θ is *twisted* if the following two conditions are satisfied:

1. θ is not constant,
2. ω is not rotationally symmetric with respect to the origin in $\mathbb{R}^2$.

Otherwise we say that Ω_θ is *untwisted*.

Here the precise meaning of ω being "rotationally symmetric with respect to the origin in $\mathbb{R}^2$" is that, for every $\vartheta \in (0, 2\pi)$,

$$\omega_\vartheta := \left\{ x_2 \cos\vartheta + x_3 \sin\vartheta, -x_2 \sin\vartheta + x_3 \cos\vartheta \,\middle|\, (x_2, x_3) \in \omega \right\} = \omega \,,$$

with the natural convention that we identify ω and ω_ϑ (and other open sets) provided that they differ on a set of zero capacity. Hence, modulus a set of zero capacity, ω is rotationally symmetric with respect to the origin in $\mathbb{R}^2$ if, and only if, it is a disc or an annulus centered at the origin of $\mathbb{R}^2$. In view of this convention, any untwisted Ω_θ can be identified with the straight tube Ω_0 by an isometry of the Euclidean space.

We write $x = (x_1, x_2, x_3)$ for a point/vector in $\mathbb{R}^3$. If x is used to denote a point in Ω_0 or Ω_θ, we refer to x_1 and $x' := (x_2, x_3)$ as "longitudinal" and "transverse" variables in the tube, respectively.

It is easy to check that the mapping $\mathcal{L}_\theta$ is injective and that its Jacobian is identically equal to 1. Consequently, $\mathcal{L}_\theta$ induces a (global) diffeomorphism between Ω_0 and Ω_θ.

2.2 The Dirichlet Laplacian in a twisted tube

It follows from the last result that Ω_θ is an open set. The corresponding Dirichlet Laplacian in $L^2(\Omega_\theta)$ can be therefore introduced in a standard way as the self-adjoint operator $-\Delta_D^{\Omega_\theta}$ associated with the quadratic form

$$Q_D^{\Omega_\theta}[\Psi] := \|\nabla\Psi\|^2_{L^2(\Omega_\theta)}, \qquad \Psi \in \mathfrak{D}(Q_D^{\Omega_\theta}) := H_0^1(\Omega_\theta) \,.$$

By the representation theorem, $-\Delta_D^{\Omega_\theta}\Psi = -\Delta\Psi$ for $\Psi \in \mathfrak{D}(-\Delta_D^{\Omega_\theta}) := \{\Psi \in H_0^1(\Omega_\theta) \,|\, \Delta\Psi \in L^2(\Omega_\theta)\}$, where the Laplacian $\Delta\Psi$ should be understood in the distributional sense.

Moreover, using the diffeomorphism induced by $\mathcal{L}_\theta$, we can "untwist" the tube by expressing the Laplacian $-\Delta_D^{\Omega_\theta}$ in the curvilinear coordinates determined by (2.1). More precisely, let U_θ be the unitary transformation from $L^2(\Omega_\theta)$ to $L^2(\Omega_0)$ defined by

$$U_\theta\Psi := \Psi \circ \mathcal{L}_\theta \,. \tag{2.2}$$

It is easy to check that $H_\theta := U_\theta(-\Delta_D^{\Omega_\theta})U_\theta^{-1}$ is the self-adjoint operator in $L^2(\Omega_0)$ associated with the quadratic form

$$Q_\theta[\psi] := \|\partial_1\psi - \dot\theta\,\partial_\tau\psi\|^2_{L^2(\Omega_0)} + \|\nabla'\psi\|^2_{L^2(\Omega_0)} \,, \qquad \psi \in \mathfrak{D}(Q_\theta) := H_0^1(\Omega_0) \,. \tag{2.3}$$

Here $\nabla' := (\partial_2, \partial_3)$ denotes the transverse gradient and ∂_τ is a shorthand for the transverse angular-derivative operator

$$\partial_\tau := \tau \cdot \nabla' = x_3\partial_2 - x_2\partial_3 \,, \qquad \text{where} \qquad \tau(x_2, x_3) := (x_3, -x_2) \,.$$

We have the point-wise estimate

$$|\partial_\tau\psi| \le a\,|\nabla'\psi| \,, \qquad \text{where} \qquad a := \sup_{x' \in \omega} |x'| \,. \tag{2.4}$$

The sesquilinear form associated with $Q_\theta[\cdot]$ will be denoted by $Q_\theta(\cdot,\cdot)$. In the distributional sense, we can write

$$H_\theta\psi = -(\partial_1 - \dot\theta\,\partial_\tau)^2\psi - \Delta'\psi \,, \tag{2.5}$$

where $-\Delta' := -\partial_2^2 - \partial_3^2$ denotes the transverse Laplacian.

3 The Hardy and Sobolev inequalities

In this section we summarize basic spectral results about the Laplacian $-\Delta_D^{\Omega_\theta}$ we shall need later to study the asymptotic behaviour of the associated semigroup.

3.1 The Poincaré inequality

Let E_1 be the first eigenvalue of the Dirichlet Laplacian in ω. Using the Poincaré-type inequality in the cross-section

$$\|\nabla f\|^2_{L^2(\omega)} \geq E_1 \|f\|^2_{L^2(\omega)}\,, \qquad \forall f \in H_0^1(\omega)\,, \tag{3.1}$$

and Fubini's theorem, it readily follows that $Q_\theta[\psi] \geq E_1 \|\psi\|^2_{L^2(\Omega_0)}$ for every $\psi \in H_0^1(\Omega_0)$. Or, equivalently,

$$-\Delta_D^{\Omega_\theta} \geq E_1 \tag{3.2}$$

in the form sense in $L^2(\Omega_\theta)$. Consequently, the spectrum of $-\Delta_D^{\Omega_\theta}$ does not start below E_1. The result (3.2) can be interpreted as a Poincaré-type inequality and it holds for any tube Ω_θ.

The inequality (3.2) is clearly sharp for an untwisted tube, since (1.4) holds in that case trivially by separation of variables. In general, the spectrum of $-\Delta_D^{\Omega_\theta}$ can start strictly above E_1 if the twisting is effective at infinity (*cf* [18, Corol. 6.6]). In this paper, however, we focus on tubes for which the energy E_1 coincides with the spectral threshold of $-\Delta_D^{\Omega_\theta}$. This is typically the case if the twisting vanishes at infinity (*cf* [17, Sec. 4]). More restrictively, we assume (1.3). Under this hypothesis, (1.4) holds and (3.2) is sharp in the twisted case too.

3.2 The Poincaré inequality in a bounded tube

For our further purposes, it is important that a better result than (3.2) holds in bounded tubes.

Given a bounded open interval $I \subset \mathbb{R}$, let H_θ^I be the "restriction" of H_θ to the tube $I \times \omega$ determined by the conditions $\partial_1\psi - \dot\theta\partial_\tau\psi = 0$ on the new boundary $(\partial I)\times\omega$. More precisely, H_θ^I is introduced as the self-adjoint operator in $L^2(I\times\omega)$ associated with the quadratic form

$$\begin{aligned} Q_\theta^I[\psi] &:= \|\partial_1\psi - \dot\theta\,\partial_\tau\psi\|^2_{L^2(I\times\omega)} + \|\nabla'\psi\|^2_{L^2(I\times\omega)}\,, \\ \psi \in \mathfrak{D}(Q_\theta^I) &:= \left\{\psi \restriction (I\times\Omega) \mid \psi \in H_0^1(\Omega_0)\right\}\,. \end{aligned}$$

That is, we impose no additional boundary conditions in the form setting.

Contrary to H_θ, H_θ^I is an operator with compact resolvent. Let $\lambda(\dot\theta, I)$ denote the lowest eigenvalue of the shifted operator $H_\theta^I - E_1$. We have the following variational characterization:

$$\lambda(\dot\theta, I) = \min_{\psi\in\mathfrak{D}(Q_\theta^I)\setminus\{0\}} \frac{Q_\theta^I[\psi] - E_1\|\psi\|^2_{L^2(I\times\omega)}}{\|\psi\|^2_{L^2(I\times\omega)}}\,. \tag{3.3}$$

As in the unbounded case, (3.1) yields that $\lambda(\dot\theta, I)$ is non-negative (it is zero if the tube is untwisted). However, thanks to the compactness, now we have that $H_\theta^I - E_1$ is a positive operator whenever the tube is twisted.

Lemma 3.1. *Let $\theta \in C^1(\mathbb{R})$. Let $I \subset \mathbb{R}$ be a bounded open interval such that $\theta \upharpoonright I$ is not constant. Let ω be not rotationally invariant with respect to the origin in $\mathbb{R}^2$. Then*

$$\lambda(\dot\theta, I) > 0\,.$$

Proof. We proceed by contradiction and assume that $\lambda(\dot\theta, I) = 0$. Then the minimum (3.3) is attained by a (smooth) function $\psi \in \mathfrak{D}(Q_\theta^I)$ satisfying (recall (3.1))

$$\|\partial_1\psi - \dot\theta\,\partial_\tau\psi\|^2_{L^2(I\times\omega)} = 0 \quad \text{and} \quad \|\nabla'\psi\|^2_{L^2(I\times\omega)} - E_1\|\psi\|^2_{L^2(I\times\omega)} = 0\,. \quad (3.4)$$

Writing $\psi(x) = \varphi(x_1)\mathcal{J}_1(x') + \phi(x)$, where $\mathcal{J}_1$ is the positive eigenfunction corresponding to E_1 of the Dirichlet Laplacian in $L^2(\omega)$ and $(\mathcal{J}_1, \phi(x_1,\cdot))_{L^2(\omega)} = 0$ for every $x_1 \in I$, we deduce from the second equality in (3.4) that $\phi = 0$. The first identity is then equivalent to

$$\|\dot\varphi\|^2_{L^2(I)}\|\mathcal{J}_1\|^2_{L^2(\omega)} + \|\dot\theta\,\varphi\|^2_{L^2(I)}\|\partial_\tau\mathcal{J}_1\|^2_{L^2(\omega)} - 2(\mathcal{J}_1, \partial_\tau\mathcal{J}_1)_{L^2(\omega)}\Re(\dot\varphi, \dot\theta\,\varphi)_{L^2(I)} = 0\,.$$

Since $(\mathcal{J}_1, \partial_\tau\mathcal{J}_1)_{L^2(\omega)} = 0$ by an integration by parts, it follows that φ must be constant and that

$$\|\dot\theta\|_{L^2(I)} = 0 \qquad \text{or} \qquad \|\partial_\tau\mathcal{J}_1\|_{L^2(\omega)} = 0\,.$$

However, this is impossible under the stated assumptions because $\|\dot\theta\|_{L^2(I)}$ vanishes if and only if θ is constant on I, and $\partial_\tau\mathcal{J}_1 = 0$ identically in ω if and only if ω is rotationally invariant with respect to the origin. ☐

Lemma 3.1 was the cornerstone of the method of [18] to establish the existence of Hardy inequalities in twisted tubes (see also the proof of Theorem 3.1 below).

3.3 Infinitesimally thin tubes

It is clear that $\lambda(\dot\theta, \mathbb{R}) := \inf\sigma(H_\theta) = 0$ whenever (1.4) holds (*e.g.*, if (1.3) is satisfied). It turns out that the shifted spectral threshold diminishes also in the opposite asymptotic regime, *i.e.* when the interval $I_\epsilon := (-\epsilon, \epsilon)$ shrinks, and this irrespectively of the properties of ω and $\dot\theta$.

Proposition 3.1. *Let $\theta \in C^1(\mathbb{R})$. We have*

$$\lim_{\epsilon\to 0} \lambda\big(\dot\theta, I_\epsilon\big) = 0\,.$$

Proof. Let $\{\omega_k\}_{k=0}^\infty$ be an exhaustion sequence of ω, *i.e.*, each ω_k is an open set with smooth boundaries satisfying $\omega_k \Subset \omega_{k+1}$ and $\cup_{k=0}^\infty \omega_k = \omega$. Let $\mathcal{J}_1^k$ denote the first eigenfunction of the Dirichlet Laplacian in $L^2(\omega_k)$; we extend it by zero to the whole $\mathbb{R}^2$. Finally, set $\psi^k := (1 \otimes \mathcal{J}_1^k) \circ \mathcal{L}_{\theta_0}$ with $\theta_0(x_1) := \dot\theta(0)\,x_1$, *i.e.*,

$$\psi^k(x) = \mathcal{J}_1^k\Big(x_2\cos(\dot\theta_0 x_1) + x_3\sin(\dot\theta_0 x_1), -x_2\sin(\dot\theta_0 x_1) + x_3\cos(\dot\theta_0 x_1)\Big)\,,$$

where $\dot\theta_0 := \dot\theta(0)$.

For any (large) $k \in \mathbb{N}$ there exists (small) positive ϵ_k such that ψ^k belongs to $\mathfrak{D}(Q_\theta^{I_\epsilon})$ for all $\epsilon \leq \epsilon_k$. Hence it is an admissible trial function for (3.3). Now, fix $k \in \mathbb{N}$ and assume that $\epsilon \leq \epsilon_k$. Then we have

$$\|\psi^k\|^2_{L^2(I_\epsilon\times\omega)} = |I_\epsilon|\, \|\mathcal{J}_1^k\|^2_{L^2(\omega_k)} ,$$

where we have used the change of variables $y = \mathcal{L}_{\theta_0}(x)$. At the same time, employing consecutively the identity $\partial_1\psi^k - \dot\theta\,\partial_\tau\psi^k = (\dot\theta_0 - \dot\theta)\,\partial_\tau\psi^k$, the bound (2.4), the identity $|\nabla'\psi^k(x)| = |\nabla\mathcal{J}_1^k(y_2, y_3)|$ and the same change of variables, we get the estimate

$$\|\partial_1\psi^k - \dot\theta\,\partial_\tau\psi^k\|^2_{L^2(I_\epsilon\times\omega)} \leq \|(\dot\theta_0 - \dot\theta)\|^2_{L^\infty(I_\epsilon)}\,|I_\epsilon|\,a^2\,\|\nabla\mathcal{J}_1^k\|^2_{L^2(\omega_k)} ,$$

where the supremum norm clearly tends to zero as $\epsilon \to 0$. Finally,

$$\|\nabla'\psi^k\|^2_{L^2(I_\epsilon\times\omega)} - E_1\|\psi^k\|^2_{L^2(I_\epsilon\times\omega)} = (E_1^k - E_1)\,|I_\epsilon|\,\|\mathcal{J}_1^k\|^2_{L^2(\omega_k)} ,$$

where E_1^k denotes the first eigenvalue of the Dirichlet Laplacian in $L^2(\omega_k)$. Sending ϵ to zero, the trial-function argument therefore yields

$$\lim_{\epsilon\to 0} \lambda(\dot\theta, I_\epsilon) \leq E_1^k - E_1 .$$

Since k can be made arbitrarily large and $E_1^k \to E_1$ as $k \to \infty$ by standard approximation arguments (see, *e.g.*, [4]), we conclude with the desired result. □

Remark 3.1 (An erratum to [17]). The study of the infinitesimally thin tubes played a crucial in the proof of Hardy inequalities given in [17]. According to Lemma 6.3 in [17], $\lambda\big(\dot\theta, I_\epsilon\big)$, with constant $\dot\theta$, is independent of $\epsilon > 0$ (and therefore remains positive for a twisted tube even if $\epsilon \to 0$). However, in view Proposition 3.1, this is false. Consequently, Lemmata 6.3 and 6.5 and Theorem 6.6 in [17] cannot hold. The proof of Hardy inequalities presented in [17] is incorrect. A corrected version of the paper [17] can be found in [18].

3.4 The Hardy inequality

Now we come back to unbounded tubes Ω_θ. Although (3.2) represents a sharp Poincaré-type inequality both for twisted and untwisted tubes (if (1.4) holds), there is a fine difference in the spectral setting. Whenever the tube Ω_θ is non-trivially twisted (*cf* Definition 2.1), there exists a positive function ϱ (necessarily vanishing at infinity if (1.4) holds) such that (3.2) is improved to (1.1). A variant of the Hardy inequality is represented by the following theorem:

Theorem 3.1. *Let $\theta \in C^1(\mathbb{R})$ and suppose that $\dot\theta$ has compact support. Then for every $\Psi \in H_0^1(\Omega_\theta)$ we have*

$$\|\nabla\Psi\|^2_{L^2(\Omega_\theta)} - E_1\,\|\Psi\|^2_{L^2(\Omega_\theta)} \;\geq\; c_H\,\|\rho\Psi\|^2_{L^2(\Omega_\theta)} , \tag{3.5}$$

where $\rho(x) := 1/\sqrt{1+x_1^2}$ and c_H is a non-negative constant depending on $\dot\theta$ and ω. Moreover, c_H is positive if, and only if, Ω_θ is twisted.

Proof. It is clear that the left hand side of (3.5) is non-negative due to (3.2). The fact that $c_H = 0$ if the tube is untwisted follows from the more general result included in Proposition 3.2.2 below. We divide the proof of the converse fact (*i.e.* that twisting implies $c_H > 0$) into several steps. Recall the identification of $\Psi \in L^2(\Omega_\theta)$ with $\psi := U_\theta \Psi \in L^2(\Omega_0)$ via (2.2).

1. Let us first assume that the interval $I := (\inf \operatorname{supp} \dot\theta, \sup \operatorname{supp} \dot\theta)$ is symmetric with respect to the origin of $\mathbb{R}$.

2. The main ingredient in the proof is the following Hardy-type inequality for a Schrödinger operator in $\mathbb{R} \times \omega$ with a characteristic-function potential:

$$\|\rho\psi\|^2_{L^2(\Omega_0)} \leq 16 \, \|\partial_1\psi\|^2_{L^2(\Omega_0)} + (2 + 64/|I|^2) \, \|\psi\|^2_{L^2(I\times\omega)} \tag{3.6}$$

for every $\psi \in H_0^1(\Omega_0)$. This inequality is a consequence of the classical one-dimensional Hardy inequality $\int_{\mathbb{R}} x_1^{-2} |\varphi(x_1)|^2 dx_1 \leq 4 \int_{\mathbb{R}} |\dot\varphi(x_1)|^2 dx_1$ valid for any $\varphi \in H_0^1(\mathbb{R}\setminus\{0\})$. Indeed, following [7, Sec. 3.3], let η be the Lipschitz function on $\mathbb{R}$ defined by $\eta(x_1) := 2|x_1|/|I|$ for $|x_1| \leq |I|/2$ and 1 otherwise in $\mathbb{R}$ (we shall denote by the symbol the function $\eta \otimes 1$ on $\mathbb{R} \times \omega$). For any $\psi \in C_0^\infty(\Omega_0)$, let us write $\psi = \eta\psi + (1-\eta)\psi$, so that $(\eta\psi)(\cdot, x') \in H_0^1(\mathbb{R}\setminus\{0\})$ for every $x' \in \omega$. Then, employing Fubini's theorem, we can estimate as follows:

$$\begin{aligned}
\|\rho\psi\|^2_{L^2(\Omega_0)} &\leq 2 \int_{\Omega_0} x_1^{-2} \, |(\eta\psi)(x)|^2 \, dx + 2 \, \|(1-\eta)\psi\|^2_{L^2(\Omega_0)} \\
&\leq 8 \, \|\partial_1(\eta\psi)\|^2_{L^2(\Omega_0)} + 2 \, \|\psi\|^2_{L^2(I\times\omega)} \\
&\leq 16 \, \|\eta\partial_1\psi\|^2_{L^2(\Omega_0)} + 16 \, \|(\partial_1\eta)\psi\|^2_{L^2(\Omega_0)} + 2 \, \|\psi\|^2_{L^2(I\times\omega)} \\
&\leq 16 \, \|\partial_1\psi\|^2_{L^2(\Omega_0)} + (2 + 64/|I|^2) \, \|\psi\|^2_{L^2(I\times\omega)} \, .
\end{aligned}$$

By density, this result extends to all $\psi \in H_0^1(\Omega_0) = \mathfrak{D}(Q_\theta)$.

3. By Lemma 3.1, we have

$$Q_\theta[\psi] - E_1 \, \|\psi\|^2_{L^2(\Omega_0)} \;\geq\; Q_\theta^I[\psi] - E_1 \, \|\psi\|^2_{L^2(I\times\omega)} \;\geq\; \lambda(\dot\theta, I) \, \|\psi\|^2_{L^2(I\times\omega)} \tag{3.7}$$

for every $\psi \in \mathfrak{D}(Q_\theta)$. Here the first inequality employs the trivial fact that the restriction to $I \times \omega$ of a function from $\mathfrak{D}(Q_\theta)$ belongs to $\mathfrak{D}(Q_\theta^I)$. Under the stated hypotheses, we know from Lemma 3.1 that $\lambda(\dot\theta, I)$ is a positive number.

4. At the same time, for every $\psi \in \mathfrak{D}(Q_\theta)$,

$$\begin{aligned}
&Q_\theta[\psi] - E_1 \, \|\psi\|^2_{L^2(\Omega_0)} \\
&\quad \geq \epsilon \, \|\partial_1\psi\|^2_{L^2(\Omega_0)} + \int_{\Omega_0} \left\{ \left[1 - \frac{\epsilon}{1-\epsilon} \, a^2 \, \dot\theta^2(x_1)\right] |\nabla'\psi(x)|^2 - E_1 \, |\psi(x)|^2 \right\} dx \\
&\quad \geq \epsilon \, \|\partial_1\psi\|^2_{L^2(\Omega_0)} - \frac{\epsilon}{1-\epsilon} \, a^2 E_1 \, \|\dot\theta\psi\|^2_{L^2(\Omega_0)} \\
&\quad \geq \epsilon \, \|\partial_1\psi\|^2_{L^2(\Omega_0)} - \frac{\epsilon}{1-\epsilon} \, \|\dot\theta\|^2_{L^\infty(I)} \, a^2 E_1 \, \|\psi\|^2_{L^2(I\times\omega)}
\end{aligned} \tag{3.8}$$

for sufficiently small positive ϵ. Here the first estimate is an elementary Cauchy-type inequality employing (2.4) and valid for all $\epsilon \in (0,1)$. The second inequality in (3.8) follows from (3.1) with help of Fubini's theorem provided that ϵ is sufficiently small, namely if $\epsilon < \left(1 + a^2 \|\dot\theta\|^2_{L^\infty(\mathbb{R})}\right)^{-1}$.

5. Interpolating between the bounds (3.7) and (3.8), and using (3.6) in the latter, we finally arrive at

$$
\begin{aligned}
Q_\theta[\psi] - E_1 \|\psi\|^2_{L^2(\Omega_0)} \geq \frac{1}{2}\frac{\epsilon}{16} \, \|\rho\psi\|^2_{L^2(\Omega_0)} \\
+ \frac{1}{2}\left[\lambda(\dot\theta, I) - \epsilon\left(\frac{1}{8}+\frac{4}{|I|^2}\right) - \frac{\epsilon}{1-\epsilon}\,\|\dot\theta\|^2_{L^\infty(I)}\, a^2 E_1\right] \|\psi\|^2_{L^2(I\times\omega)}
\end{aligned}
$$

for every $\psi \in \mathfrak{D}(Q_\theta)$. It is clear that the last line on the right hand side of this inequality can be made non-negative by choosing ϵ sufficiently small. Such an ϵ then determines the Hardy constant $c_H' := \epsilon/32$.

6. The previous bound can be transferred to $L^2(\Omega_\theta)$ via (2.2). In general, if the centre of I is an arbitrary point $x_1^0 \in \mathbb{R}$, the obtained result is equivalent to

$$
\forall \Psi \in \mathfrak{D}(Q_D^{\Omega_\theta}) \,, \qquad \|\nabla\Psi\|^2_{L^2(\Omega_\theta)} - E_1 \,\|\Psi\|^2_{L^2(\Omega_\theta)} \;\geq\; c_H' \,\|\rho_{x_1^0}\Psi\|^2_{L^2(\Omega_\theta)} \,,
$$

where $\rho_{x_1^0}(x) := 1/\sqrt{1+(x_1-x_1^0)^2}$. This yields (3.5) with

$$
c_H := c_H' \min_{x_1\in\mathbb{R}} \frac{1+x_1^2}{1+(x_1-x_1^0)^2} \,,
$$

where the minimum is a positive constant depending on x_1^0. □

The Hardy inequality of Theorem 3.1 was first established [7] under additional hypotheses. The present version is adopted from [18], where other variants of the inequality can be found, too.

3.5 The spectral stability

Theorem 3.1 provides certain stability properties of the spectrum for twisted tubes, while the untwisted case is always unstable, in the following sense:

Proposition 3.2. *Let V be the multiplication operator in $L^2(\Omega_\theta)$ by a bounded non-zero non-negative function v such that $v(x) \sim |x_1|^{-2}$ as $|x_1| \to \infty$. Then*

1. *if Ω_θ is twisted with $\theta \in C^1(\mathbb{R})$ and $\dot\theta$ has compact support, then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,*

$$
\inf \sigma(-\Delta_D^{\Omega_\theta} - \varepsilon V) \geq E_1 \,;
$$

2. *if Ω_θ is untwisted then, for all $\varepsilon > 0$,*

$$
\inf \sigma(-\Delta_D^{\Omega_\theta} - \varepsilon V) < E_1 \,.
$$

Proof. The first statement follows readily from one part of Theorem 3.1. To prove the second property (and therefore the other part of Theorem 3.1 stating that $c_H = 0$ if the tube is untwisted), it is enough to consider the case $\theta = 0$ and construct a test function ψ from $H_0^1(\Omega_0)$ such that

$$
P_0[\psi] := \|\nabla\psi\|^2_{L^2(\Omega_0)} - E_1\|\psi\|^2_{L^2(\Omega_0)} - \varepsilon \left\|v^{1/2}\psi\right\|^2_{L^2(\Omega_0)} < 0
$$

for all positive ε. For every $n \geq 1$, we define

$$\psi_n(x) := \varphi_n(x_1)\mathcal{J}_1(x_2, x_3)\,, \tag{3.9}$$

where $\mathcal{J}_1$ is the positive eigenfunction corresponding to E_1 of the Dirichlet Laplacian in the cross-section ω, normalized to 1 in $L^2(\omega)$, and

$$\varphi_n(x_1) := \exp\left(-\frac{x_1^2}{n}\right). \tag{3.10}$$

In view of the separation of variables and the normalization of $\mathcal{J}_1$, we have

$$P_0[\psi_n] = \|\dot{\varphi}_n\|^2_{L^2(\mathbb{R})} - \varepsilon \left\|v_1^{1/2}\varphi_n\right\|^2_{L^2(\mathbb{R})}\,,$$

where $v_1(x_1) := \|v(x_1,\cdot)^{1/2}\mathcal{J}_1\|^2_{L^2(\omega)}$. By hypothesis, $v_1 \in L^1(\mathbb{R})$ and the integral $\|v_1\|_{L^1(\mathbb{R})}$ is positive. Finally, an explicit calculation yields $\|\dot{\varphi}_n\|_{L^2(\mathbb{R})} \sim n^{-1/4}$. By the dominated convergence theorem, we therefore have

$$P_0[\psi_n] \xrightarrow[n\to\infty]{} -\varepsilon \,\|v_1\|_{L^1(\mathbb{R})}\,.$$

Consequently, taking n sufficiently large and ε positive, we can make the form $P_0[\psi_n]$ negative. □

Since the potential V is bounded and vanishes at infinity, it is easy to see that the essential spectrum is not changed, *i.e.*, $\sigma_{\mathrm{ess}}(-\Delta_D^{\Omega_\theta} - \varepsilon V) = [E_1, \infty)$, independently of the value of ε and irrespectively of whether the tube is twisted or not. As a consequence of Proposition 3.2, we have that an arbitrarily small attractive potential $-\varepsilon V$ added to the shifted operator $-\Delta_D^{\Omega_\theta} - E_1$ in the untwisted tube would generate negative discrete eigenvalues, however, a certain critical value of ε is needed in order to generate the negative spectrum in the twisted case. In the language of [21], the operator $-\Delta_D^{\Omega_\theta} - E_1$ is therefore subcritical (respectively critical) if Ω_θ is twisted (respectively untwisted).

3.6 An upper bound to the Hardy constant

Now we come back to Theorem 3.1 and show that the Hardy weight at the right hand side of (3.5) cannot be made arbitrarily large by increasing $\dot{\theta}$ or making the cross-section ω more eccentric.

Proposition 3.3. *Let $\theta \in C^1(\mathbb{R})$ and suppose that $\dot{\theta}$ has compact support. Then*

$$c_H \leq 1/2\,,$$

where c_H is the constant of Theorem 3.1.

Proof. Recall the unitary equivalence of $-\Delta_D^{\Omega_\theta}$ and H_θ given by (2.2). We proceed by contradiction and show that the operator $H_\theta - E_1 - c\rho^2$ is not non-negative if $c > 1/2$, irrespectively of properties of θ and ω. (Recall that ρ was initially introduced in Theorem 3.1 as a function on Ω_θ. In this proof, with an abuse of notation, we denote by the same symbol analogous functions on Ω_0 and $\mathbb{R}$.) It is enough to construct a test function ψ from $\mathfrak{D}(Q_\theta)$ such that

$$P_\theta^c[\psi] := Q_\theta[\psi] - E_1\|\psi\|^2_{L^2(\Omega_0)} - c\,\|\rho\psi\|^2_{L^2(\Omega_0)} < 0\,.$$

As in the proof of Proposition 3.2, we use the decomposition (3.9), but now the sequence of functions φ_n is defined as follows:

$$\varphi_n(x_1) := \begin{cases} \frac{x_1-b_n^1}{b_n^2-b_n^1} & \text{if} \quad x_1 \in [b_n^1, b_n^2)\,, \\ \frac{b_n^3-x_1}{b_n^3-b_n^2} & \text{if} \quad x_1 \in [b_n^2, b_n^3)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Here $\{b_n^j\}_{n\in\mathbb{N}}$, with $j = 1, 2, 3$, are numerical sequences such that $\sup \operatorname{supp} \dot{\theta} < b_n^1 < b_n^2 < b_n^3$ for each $n \in \mathbb{N}$ and $b_n^1 \to \infty$ as $n \to \infty$; further requirements will be imposed later on. Since φ_n and $\dot{\theta}$ have disjoint supports, and $\mathcal{J}_1$ is supposed to be normalized to 1 in $L^2(\omega)$, it easily follows that

$$P_\theta^c[\psi_n] = \|\dot{\varphi}_n\|^2_{L^2(\mathbb{R})} - c\,\|\rho\varphi_n\|^2_{L^2(\mathbb{R})}\,.$$

Note that the right hand side is independent of θ and ω. An explicit calculation yields

$$\begin{aligned} \|\dot{\varphi}_n\|^2_{L^2(\mathbb{R})} &= \frac{1}{b_n^2-b_n^1} + \frac{1}{b_n^3-b_n^2}\,, \\ \|\rho\varphi_n\|^2_{L^2(\mathbb{R})} &= \frac{b_n^2-b_n^1+[(b_n^1)^2-1](\arctan b_n^2-\arctan b_n^1)-b_n^1\log\frac{1+(b_n^2)^2}{1+(b_n^1)^2}}{(b_n^2-b_n^1)^2} \\ &\quad + \frac{b_n^3-b_n^2+[(b_n^3)^2-1](\arctan b_n^3-\arctan b_n^2)-b_n^3\log\frac{1+(b_n^3)^2}{1+(b_n^2)^2}}{(b_n^3-b_n^2)^2}\,. \end{aligned}$$

Specifying the numerical sequences in such a way that also the quotients b_n^2/b_n^1 and b_n^3/b_n^2 tend to infinity as $n \to \infty$, it is then straightforward to check that

$$b_n^2\,P_\theta^c[\psi_n] \xrightarrow[n\to\infty]{} 1-2c\,.$$

Since the limit is negative for $c > 1/2$, it follows that $P_\theta^c[\psi_n]$ can be made negative by choosing n sufficiently large. $\square$

The proposition shows that the effect of twisting is limited in its nature, at least if (1.3) holds. This will have important consequences for the usage of energy methods when studying the heat semigroup below.

3.7 The Sobolev inequality

Regardless of whether the tube is twisted or not, the operator $-\Delta_D^{\Omega_\theta} - E_1$ satisfies the following Sobolev-type inequality.

Theorem 3.2 (Sobolev inequality)**.** *Let $\theta \in C^1(\mathbb{R})$ and suppose that $\dot{\theta}$ has compact support. Then for every $\Psi \in H_0^1(\Omega_\theta) \cap L^2(\Omega_\theta, \rho^{-2})$ we have*

$$\|\nabla\Psi\|^2_{L^2(\Omega_\theta)} - E_1\|\Psi\|^2_{L^2(\Omega_\theta)} \;\geq\; c_S\,\frac{\|\Psi\|^6_{L^2(\Omega_\theta)}}{\|\Psi\|_1^4}\,, \tag{3.11}$$

where $\|\Psi\|_1 := \sqrt{\int_\omega dx_2 dx_3 \left(\int_\mathbb{R} dx_1 |(\Psi\circ\mathcal{L}_\theta)(x)|\right)^2}$ and c_S is a positive constant depending on $\dot{\theta}$ and ω.

Proof. Recall that $\Psi \circ \mathcal{L}_\theta = U_\theta \Psi =: \psi$ belongs to $L^2(\Omega_0)$. First of all, let us notice that $\|\Psi\|_1$ is well defined for $\Psi \in L^2(\Omega_\theta, \rho^{-2})$. Indeed, the Schwarz inequality together with Fubini's theorem yields

$$\|\Psi\|_1^2 \leq \|\rho^{-1}\psi\|_{L^2(\Omega_0)}^2 \int_{\mathbb{R}} \frac{dx_1}{1+x_1^2} = \|\rho^{-1}\Psi\|_{L^2(\Omega_\theta)}^2 \, \pi < \infty \, . \tag{3.12}$$

Here the equality of the norms is obvious from the facts that the mapping $\mathcal{L}_\theta$ leaves invariant the first coordinate in $\mathbb{R}^3$ and that its Jacobian is one. We also remark that, by density, it is enough to prove the theorem for $\Psi \in C_0^\infty(\Omega_\theta)$.

The inequality (3.11) is a consequence of the one-dimensional inequality

$$\forall \varphi \in H^1(\mathbb{R}) \cap L^1(\mathbb{R}) \, , \qquad \|\dot{\varphi}\|_{L^2(\mathbb{R})}^2 \geq \frac{1}{4} \, \frac{\|\varphi\|_{L^2(\mathbb{R})}^6}{\|\varphi\|_{L^1(\mathbb{R})}^4} \, , \tag{3.13}$$

which is established quite easily by combining elementary estimates

$$\|\varphi\|_{L^2(\mathbb{R})}^2 \leq \|\varphi\|_{L^1(\mathbb{R})} \|\varphi\|_{L^\infty(\mathbb{R})} \qquad \text{and} \qquad \|\varphi\|_{L^\infty(\mathbb{R})}^2 \leq 2 \, \|\varphi\|_{L^2(\mathbb{R})} \|\dot{\varphi}\|_{L^2(\mathbb{R})} \, .$$

In order to apply (3.13), we need to estimate the left hand side of (3.11) from below by $\|\partial_1 \psi\|_{L^2(\Omega_0)}^2$. We can proceed as in the proof of Theorem 3.1. Interpolating between the bounds (3.7) and (3.8), we get

$$\|\nabla \Psi\|_{L^2(\Omega_\theta)}^2 - E_1 \|\Psi\|_{L^2(\Omega_\theta)}^2 \geq \frac{\epsilon}{2} \, \|\partial_1 \psi\|_{L^2(\Omega_0)}^2 \, ,$$

where $\epsilon =: 8\, c_S$ is a positive constant depending on $\dot{\theta}$ and ω. Using now (3.13) with help of Fubini's theorem, we conclude the proof with

$$\|\partial_1 \psi\|_{L^2(\Omega_0)}^2 \geq \frac{1}{4} \int_\omega \frac{\|\psi(\cdot, x_2, x_3)\|_{L^2(\mathbb{R})}^6}{\|\psi(\cdot, x_2, x_3)\|_{L^1(\mathbb{R})}^4} \, dx_2 dx_3 \geq \frac{1}{4} \frac{\|\Psi\|_{L^2(\Omega_\theta)}^6}{\|\Psi\|_1^4} \, .$$

Here the second inequality follows by the Hölder inequality with properly chosen conjugate exponents (recall also that $\|\psi\|_{L^2(\Omega_0)} = \|\Psi\|_{L^2(\Omega_\theta)}$). □

4 The energy estimates

4.1 The heat equation

Having the replacement $u(x,t) \mapsto e^{-E_1 t} \, u(x,t)$ for (1.2) in mind, let us consider the following t-time evolution problem in the tube Ω_θ:

$$\begin{cases} u_t - \Delta u - E_1 u = 0 & \text{in} \quad \Omega_\theta \times (0,\infty) \, , \\ u = u_0 & \text{in} \quad \Omega_\theta \times \{0\} \, , \\ u = 0 & \text{in} \quad (\partial\Omega_\theta) \times (0,\infty) \, , \end{cases} \tag{4.1}$$

where $u_0 \in L^2(\Omega_\theta)$.

As usual, we consider the weak formulation of the problem, *i.e.*, we say a Hilbert space-valued function $u \in L^2_{\mathrm{loc}}\big((0,\infty); H_0^1(\Omega_\theta)\big)$, with the weak derivative $u' \in L^2_{\mathrm{loc}}\big((0,\infty); H^{-1}(\Omega_\theta)\big)$, is a (global) solution of (4.1) provided that

$$\big\langle v, u'(t) \big\rangle + \big(\nabla v, \nabla u(t)\big)_{L^2(\Omega_\theta)} - E_1 \, \big(v, u(t)\big)_{L^2(\Omega_\theta)} = 0 \tag{4.2}$$

for each $v \in H_0^1(\Omega_\theta)$ and a.e. $t \in [0,\infty)$, and $u(0) = u_0$. Here $\langle\cdot,\cdot\rangle$ denotes the pairing of $H_0^1(\Omega_\theta)$ and $H^{-1}(\Omega_\theta)$. With an abuse of notation, we denote by the same symbol u both the function on $\Omega_\theta \times (0,\infty)$ and the mapping $(0,\infty) \to H_0^1(\Omega_\theta)$.

Standard semigroup theory implies that there indeed exists a unique solution of (4.1) that belongs to $C^0\big([0,\infty), L^2(\Omega_\theta)\big)$. More precisely, the solution is given by $u(t) = S(t)u_0$, where $S(t)$ is the heat semigroup (1.7) associated with $-\Delta_D^{\Omega_\theta} - E_1$. By the Beurling-Deny criterion, $S(t)$ is positivity-preserving for all $t \geq 0$.

Since E_1 corresponds to the threshold of the spectrum of $-\Delta_D^{\Omega_\theta}$ if (1.3) holds, we cannot expect a uniform decay of solutions of (4.1) as $t \to \infty$ in this case. More precisely, the spectral mapping theorem together with (1.4) yields:

Proposition 4.1. *Let $\theta \in C^1(\mathbb{R})$ and suppose that $\dot\theta$ has compact support. Then for each time $t \geq 0$ we have*

$$\|S(t)\|_{L^2(\Omega_\theta)\to L^2(\Omega_\theta)} \;=\; 1\,.$$

Consequently, for each $t > 0$ and each $\varepsilon \in (0,1)$ we can find an initial datum $u_0 \in H_0^1(\Omega_\theta)$ such that $\|u_0\|_{L^2(\Omega_\theta)} = 1$ and such that the solution of (4.1) satisfies

$$\|u(t)\|_{L^2(\Omega_\theta)} \;\geq\; 1 - \varepsilon\,.$$

4.2 The dimensional decay rate

However, if we restrict ourselves to initial data decaying sufficiently fast at the infinity of the tube, it is possible to obtain a polynomial decay rate for the solutions of (4.1). In particular, we have the following result based on Theorem 3.2:

Theorem 4.1. *Let $\theta \in C^1(\mathbb{R})$ and suppose that $\dot\theta$ has compact support. Then for each time $t \geq 0$ we have*

$$\|S(t)\|_{L^2(\Omega_\theta,\rho^{-2})\to L^2(\Omega_\theta)} \;\leq\; \left(1 + \frac{4\,c_S}{\pi^2}\,t\right)^{-1/4}\,,$$

where c_S is the positive constant of Theorem 3.2 and ρ is introduced in Theorem 3.1.

Proof. The statement is equivalent to the following bound for the solution u of (4.1):

$$\forall t \in [0,\infty)\,, \qquad \|u(t)\|_{L^2(\Omega_\theta)} \;\leq\; \|\rho^{-1}u_0\|_{L^2(\Omega_\theta)} \left(1 + \frac{4\,c_S}{\pi^2}\,t\right)^{-1/4}\,, \tag{4.3}$$

where $u_0 \in L^2(\Omega_\theta,\rho^{-2})$ is any non-trivial datum. It is easy to see that the real and imaginary parts of the solution of (4.1) evolve separately. Furthermore, since $S(t)$ is positivity-preserving, given a non-negative datum u_0, the solution $u(t)$ remains non-negative for all $t \geq 0$. Consequently, establishing the bound for positive and negative parts of $u(t)$ separately, it is enough to prove (4.3) for non-negative (and non-trivial) initial data only. Without loss of generality, we therefore assume in the proof below that $u(t) \geq 0$ for all $t \geq 0$.

Let $\{\varphi_n\}_{n=1}^{\infty}$ be the family of mollifiers on $\mathbb{R}$ given by (3.10); we denote by the same symbol the functions $\varphi_n \otimes 1$ on $\mathbb{R} \times \mathbb{R}^2 \supset \Omega_\theta$. Inserting the trial function

$$v_n(x;t) := \varphi_n(x_1)\, \bar{u}_n(x_2,x_3;t)\,, \qquad \bar{u}_n(x_2,x_3;t) := \big\|\varphi_n u(\cdot,x_2,x_3;t)\big\|_{L^1(\mathbb{R})}\,,$$

into (4.2), we arrive at (recall the definition of $\|\cdot\|_1$ from Theorem 3.2)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\varphi_n u(t)\|_1^2 &= -\|\nabla \bar{u}_n(t)\|_{L^2(\omega)}^2 + E_1\|\bar{u}_n(t)\|_{L^2(\omega)}^2 - \big(\partial_1 v_n(t), \partial_1 u(t)\big)_{L^2(\Omega_\theta)} \\ &\leq \big(\partial_1 v_n(t), \partial_1 u(t)\big)_{L^2(\Omega_\theta)} \\ &\leq \|\partial_1 v_n(t)\|_{L^2(\Omega_\theta)} \|\nabla u(t)\|_{L^2(\Omega_\theta)}\,.\end{aligned}$$

Here the first inequality is due to the Poincaré-type inequality in the cross-section (3.1). We clearly have

$$\|\partial_1 v_n(t)\|_{L^2(\Omega_\theta)} = \|\dot{\varphi}_n\|_{L^2(\mathbb{R})}\, \|\bar{u}_n(t)\|_{L^2(\omega)} = \|\dot{\varphi}_n\|_{L^2(\mathbb{R})}\, \|\varphi_n u(t)\|_1\,.$$

Integrating the differential inequality, we therefore get

$$\|\varphi_n u(t)\|_1 - \|\varphi_n u_0\|_1 \leq \|\dot{\varphi}_n\|_{L^2(\mathbb{R})} \int_0^t \|\nabla u(t')\|_{L^2(\Omega_\theta)}^2\, dt'\,.$$

Since $\|\dot{\varphi}_n\|_{L^2(\mathbb{R})} \to 0$ and $\{\varphi_n\}_{n=1}^{\infty}$ is an increasing sequence of functions converging pointwise to 1 as $n \to \infty$, we conclude from this inequality that

$$\forall t \in [0,\infty)\,, \qquad \|u(t)\|_1 \leq \|u_0\|_1\,, \tag{4.4}$$

where $\|u_0\|_1$ is finite due to (3.12).

Now, substituting u for the trial function v in (4.2), applying Theorem 3.2 and using (4.4), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2(\Omega_\theta)}^2 &= -\Big(\|\nabla u(t)\|_{L^2(\Omega_\theta)}^2 - E_1\|u(t)\|_{L^2(\Omega_\theta)}^2\Big) \\ &\leq -c_S\, \frac{\|u(t)\|_{L^2(\Omega_\theta)}^6}{\|u(t)\|_1^4} \\ &\leq -c_S\, \frac{\|u(t)\|_{L^2(\Omega_\theta)}^6}{\|u_0\|_1^4}\,.\end{aligned}$$

An integration of this differential inequality leads to

$$\forall t \in [0,\infty)\,, \qquad \|u(t)\|_{L^2(\Omega_\theta)} \;\leq\; \|u_0\|_{L^2(\Omega_\theta)} \left(1 + 4\, c_S\, \frac{\|u_0\|_{L^2(\Omega_\theta)}^4}{\|u_0\|_1^4}\; t\right)^{-1/4}.$$

Dividing the last inequality by $\|\rho^{-1}u_0\|_{L^2(\Omega_\theta)}$ and replacing $\|u_0\|_1$ with $\|\rho^{-1}u_0\|_{L^2(\Omega_\theta)}$ using (3.12), we get

$$\frac{\|u(t)\|_{L^2(\Omega_\theta)}}{\|\rho^{-1}u_0\|_{L^2(\Omega_\theta)}} \;\leq\; \xi \left(1 + \frac{4\,c_S}{\pi^2}\,\xi^4\; t\right)^{-1/4} \leq \left(1 + \frac{4\,c_S}{\pi^2}\; t\right)^{-1/4},$$

where $\xi := \|u_0\|_{L^2(\Omega_\theta)}/\|\rho^{-1}u_0\|_{L^2(\Omega_\theta)} \in (0,1)$. This establishes (4.3). ☐

As a direct consequence of the theorem, we get:

Corollary 4.1. *Under the hypotheses of Theorem 4.1,* $\Gamma(\Omega_\theta) \geq 1/4$.

Proof. It is enough to realize that $L^2(\Omega_\theta, K)$ is embedded in $L^2(\Omega_\theta, \rho^{-2})$. □

The following proposition shows that the decay rate of Theorem 4.1 is optimal for untwisted tubes.

Proposition 4.2. *Let* Ω_θ *be untwisted. Then for each time* $t \geq 0$ *we have*

$$\|S(t)\|_{L^2(\Omega_\theta,K)\to L^2(\Omega_\theta)} \;\geq\; \frac{1}{\sqrt{2}}\,(1+t)^{-1/4}\;.$$

Proof. Without loss of generality, we may assume $\theta = 0$. It is enough to find an initial datum $u_0 \in L^2(\Omega_0, K)$ such that the solution u of (4.1) satisfies

$$\forall t \in [0,\infty)\,, \qquad \frac{\|u(t)\|_{L^2(\Omega_0)}}{\|u_0\|_{L^2(\Omega_0,K)}} \;\geq\; \frac{1}{\sqrt{2}}\,(1+t)^{-1/4}\;. \tag{4.5}$$

The idea is to take $u_0 := \psi_n$, where $\{\psi_n\}_{n=1}^\infty$ is the sequence (3.9) approximating a generalized eigenfunction of $-\Delta_D^{\Omega_0}$ corresponding to the threshold energy E_1. Using the fact that Ω_0 is a cross-product of $\mathbb{R}$ and ω, (4.1) can be solved explicitly in terms of an expansion into the eigenbasis of the Dirichlet Laplacian in the cross-section and a partial Fourier transform in the longitudinal variable. In particular, for our initial data we get

$$\|u(t)\|^2_{L^2(\Omega_0)} = \int_{\mathbb{R}} |\hat{\varphi}_n(\xi)|^2\, \exp\left(-2\xi^2 t\right) d\xi = \sqrt{\frac{n}{n+4t}}\sqrt{\frac{\pi n}{2}}\,,$$

where the second equality is a result of an explicit calculation enabled due to the special form of φ_n given by (3.10). At the same time, for every $n < 8$ ψ_n belongs to $L^2(\Omega_0, K)$ and an explicit calculation yields

$$\|u_0\|^2_{L^2(\Omega_0,K)} = 2\sqrt{\frac{\pi n}{8-n}}\,.$$

For the special choice $n = 6$ we get that the left hand side of (4.5) actually equals the right hand side with t being replaced by $2t/3 < t$. □

The power $1/4$ in the decay bounds of Theorem 4.1 and Proposition 4.2 reflects the quasi-one-dimensional nature of Ω_θ (*cf* (1.10)), at least if the tube is untwisted. More precisely, Proposition 4.2 readily implies that the inequality of Corollary 4.1 is sharp for untwisted tubes.

Corollary 4.2. *Let* Ω_θ *be untwisted. Then* $\Gamma(\Omega_\theta) = 1/4$.

This result establishes one part of Theorem 1.1. The much more difficult part is to show that the decay rate is improved whenever the tube is twisted.

4.3 The failure of the energy method

As a consequence of combination of direct energy arguments with Theorem 3.1, we get the following result. In Remark 4.2 below we explain why it is useless.

Theorem 4.2. *Let Ω_θ be twisted with $\theta \in C^1(\mathbb{R})$. Suppose that $\dot\theta$ has compact support. Then for each time $t \geq 0$ we have*

$$\|S(t)\|_{L^2(\Omega_\theta,\rho^{-2})\to L^2(\Omega_\theta)} \leq (1+2\,t)^{-\min\{1/2,c_H/2\}} \quad , \tag{4.6}$$

where c_H is the positive constant of Theorem 3.1.

Proof. For any positive integer n and $x \in \Omega_\theta$, let us set $\rho_n(x) := \min\{\rho(x), n^{-1}\}$. Then $\{\rho_n^{-1}\}_{n=1}^\infty$ is a non-decreasing sequence of bounded functions converging pointwise to ρ^{-1} as $n \to \infty$. Recalling the definition of ρ from Theorem 3.1, it is clear that $x \mapsto \rho_n(x)$ is in fact independent of the transverse variables x'. Moreover, $\rho_n^{-\gamma}u$ belongs to $H_0^1(\Omega_\theta)$ for every $\gamma \in \mathbb{R}$ provided $u \in H_0^1(\Omega_\theta)$.

Choosing $v := \rho_n^{-2}u$ in (4.2) (and possibly combining with the conjugate version of the equation if we allow non-real initial data), we get the identity

$$\frac{1}{2}\frac{d}{dt}\|\rho_n^{-1}u(t)\|^2 = -\|\rho_n^{-1}\nabla u(t)\|^2 + E_1\|\rho_n^{-1}u(t)\|^2 - \Re\Big(u(t)\nabla\rho_n^{-2}, \nabla u(t)\Big)\,. \tag{4.7}$$

Here and in the rest of the proof, $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and inner product in $L^2(\Omega_\theta)$ (we suppress the subscripts in this proof). Since ρ_n depends on the first variable only, we clearly have $\nabla(\rho_n^{-2}) = (-2\rho^{-3}\partial_1\rho, 0, 0)$. Introducing an auxiliary function $v_n(t) := \rho_n^{-1}u(t)$, one finds

$$\|\rho_n^{-1}\nabla u(t)\|^2 = \|\nabla v_n(t)\|^2 + \|(\partial_1\rho_n/\rho_n)\,v_n(t)\|^2 + 2\Re\Big(v_n(t), (\partial_1\rho_n/\rho_n)\,\partial_1 v_n(t)\Big)\,,$$
$$\Re\Big(u(t)\nabla\rho_n^{-2}, \nabla u(t)\Big) = -2\|(\partial_1\rho_n/\rho_n)\,v_n(t)\|^2 - 2\Re\Big(v_n(t), (\partial_1\rho_n/\rho_n)\,\partial_1 v_n(t)\Big)\,.$$

Combining these two identities and substituting the explicit expression for ρ, we see that the right hand side of (4.7) equals

$$\begin{aligned}
&-\|\nabla v_n(t)\|^2 + E_1\|v_n(t)\|^2 + \|(\partial_1\rho_n/\rho_n)\,v_n(t)\|^2 \\
&\quad = -\|\nabla v_n(t)\|^2 + E_1\|v_n(t)\|^2 + \|\chi_n\rho v_n(t)\|^2 - \|\chi_n\rho^2 v_n(t)\|^2 \\
&\quad \leq (1-c_H)\,\|\chi_n\rho v_n(t)\|^2 - \|\chi_n\rho^2 v_n(t)\|^2\,.
\end{aligned} \tag{4.8}$$

Here χ_n denotes the characteristic function of the set $\Omega_\theta^n := \Omega_\theta \cap \{\mathrm{supp}(\partial_1\rho_n)\}$, and the inequality follows from Theorem 3.1 and an obvious inclusion $\Omega_\theta^n \subset \Omega_\theta$. Substituting back the solution $u(t)$, we finally arrive at

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\rho_n^{-1}u(t)\|^2 &\leq (1-c_H)\,\|\chi_n\rho v_n(t)\|^2 - \|\chi_n\rho^2 v_n(t)\|^2 \\
&\leq (1-c_H)\,\|\chi_n\rho v_n(t)\|^2\,.
\end{aligned} \tag{4.9}$$

Now, using the monotone convergence theorem and recalling the initial data to which we restrict in the hypotheses of the theorem, the last estimate implies that $u(t)$ belongs to $L^2(\Omega_\theta, \rho^{-2})$ and that it remains true after passing to the limit $n \to \infty$, *i.e.*,

$$\frac{1}{2}\frac{d}{dt}\|\rho^{-1}u(t)\|^2 \leq (1-c_H)\,\|u(t)\|^2\,. \tag{4.10}$$

At the same time, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 &= -\Big(\|\nabla u(t)\|^2 - E_1\|u(t)\|^2\Big) \\ &\leq -c_H\,\|\rho u(t)\|^2 \\ &\leq -c_H\,\frac{\|u(t)\|^4}{\|\rho^{-1}u(t)\|^2}\,, \end{aligned} \tag{4.11}$$

where the equality follows from (4.1), the first inequality follows from Theorem 3.1 and the last inequality is established by means of the Schwarz inequality.

Summing up, in view of (4.11) and (4.10), $a(t) := \|u(t)\|^2$ and $b(t) := \|\rho^{-1}u(t)\|^2$ satisfy the system of differential inequalities

$$\dot a \leq -2\,c_H\,\frac{a^2}{b}\,, \qquad \dot b \leq 2\,(1-c_H)\,a\,, \tag{4.12}$$

with the initial conditions $a(0) = \|u_0\|^2 =: a_0$ and $b(0) = \|\rho^{-1}u_0\|^2 =: b_0$. We distinguish two cases:

1. $\underline{c_H \geq 1}$. In this case, it follows from the second inequality of (4.12) that b is decreasing. Solving the first inequality of (4.12) with b being replaced by b_0, we then get

$$a(t) \leq a_0\,\big[1 + 2\,c_H\,(a_0/b_0)\,t\big]^{-1}\,.$$

Dividing this inequality by b_0 and maximizing the resulting right hand side with respect to $a_0/b_0 \in (0,1)$, we finally get

$$\forall t \in [0,\infty)\,, \qquad \|u(t)\| \;\leq\; \|\rho^{-1}u_0\|\,(1+2\,c_H\,t)^{-1/2}\quad, \tag{4.13}$$

which in particular implies (4.6).

2. $\underline{c_H \leq 1}$. We "linearize" (4.12) by replacing one a of the square on the right hand side of first inequality by employing the second inequality of (4.12):

$$\frac{\dot a}{a} \leq -2\,c_H\,\frac{a}{b} \leq -\frac{c_H}{1-c_H}\,\frac{\dot b}{b}\,.$$

This leads to

$$a/a_0 \leq (b/b_0)^{-\frac{c_H}{1-c_H}}\,.$$

Using this estimate in the original, non-linearized system (4.12), *i.e.* solving the system by eliminating b from the first and a from the second inequality of (4.12), we respectively obtain

$$a(t) \leq a_0\,\big[1 + 2\,(a_0/b_0)\,t\big]^{-c_H}\,, \qquad b(t) \leq b_0\,\big[1 + 2\,(a_0/b_0)\,t\big]^{1-c_H}\,.$$

Dividing the first inequality by b_0 and maximizing the resulting right hand side with respect to $a_0/b_0 \in (0,1)$, we finally get

$$\forall t \in [0,\infty)\,, \qquad \|u(t)\| \;\leq\; \|\rho^{-1}u_0\|\,(1+2\,t)^{-c_H/2}\quad, \tag{4.14}$$

which is equivalent to (4.6). $\square$

Remark 4.1. We see that the power in the polynomial decay rate of Theorem 4.2 diminishes as $c_H \to 0$. Let us now argue that this cannot be improved by the present method of proof. Indeed, the first inequality of (4.11) is an application of the Hardy inequality of Theorem 3.1 and the second one is sharp. The Hardy inequality is also applied in the first inequality of (4.9). In the second inequality of (4.9), however, we have neglected a negative term. Applying the second inequality of (4.11) to it instead, we conclude with an improved system of differential inequalities

$$\dot{a} \leq -2\, c_H\, \frac{a^2}{b}\,, \qquad \dot{b} \leq 2\,(1-c_H)\, a - 2\, \frac{a^2}{b}\,. \tag{4.15}$$

The corresponding system of differential equations has the explicit solution

$$\tilde{a}(t) = a_0 \left(\frac{\xi_0}{W\big[\xi_0 \exp\left(\xi_0 + 2t\right)\big]} \right)^{c_H}, \quad \tilde{b}(t) = a(t)\Big(1 + W\big[\xi_0 \exp\left(\xi_0 + 2t\right)\big]\Big),$$

where $\xi_0 := b_0/a_0 - 1 > 0$ and W denotes the Lambert W function (product log), *i.e.* the inverse function of $w \mapsto w \exp(w)$. Since

$$W\big[\xi_0 \exp\left(\xi_0 + 2t\right)\big] = 2\, t + o(t) \qquad \text{as} \qquad t \to \infty\,,$$

we see that the $t^{-c_H/2}$ decay in (4.6) for $c_H < 1$ cannot be improved by replacing (4.12) with (4.15).

Remark 4.2. Note that the hypothesis (1.3) is not explicitly used in the proof of Theorem 4.2, it is just required that the inequality (3.5) holds with some positive constant c_H. For tubes satisfying (1.3), however, we know from Proposition 3.3 that the constant cannot exceed the value $1/2$. Consequently, irrespectively of the strength of twisting, Theorem 4.2 never represents an improvement upon Theorem 4.1. This is what we mean by the failure of a direct energy argument based on the Hardy inequality of Theorem 3.1.

5 The self-similarity transformation

Let us now turn to a completely different approach which leads to an improved decay rate regardless of the smallness of twisting.

5.1 Straightening of the tube

First of all, we reconsider the heat equation (4.1) in an untwisted tube Ω_0 by using the change of variables defined by the mapping $\mathcal{L}_\theta$. In view of the unitary transform (2.2), one can identify the Dirichlet Laplacian in $L^2(\Omega_\theta)$ with the operator (2.5) in $L^2(\Omega_0)$, and it is readily seen that (4.1) is equivalent to

$$u_t + H_\theta u - E_1 u = 0 \qquad \text{in} \qquad \Omega_0 \times (0,\infty)\,,$$

plus the Dirichlet boundary conditions on $\partial\Omega_0$ and an initial condition at $t = 0$. (We keep the same latter u for the solutions transformed to Ω_0.) More precisely, the weak formulation (4.2) is equivalent to

$$\big\langle v, u'(t) \big\rangle + Q_\theta\big(v, u(t)\big) - E_1 \big(v, u(t)\big)_{L^2(\Omega_0)} = 0 \tag{5.1}$$

for each $v \in H_0^1(\Omega_0)$ and a.e. $t \in [0,\infty)$, with $u(0) = u_0 \in L^2(\Omega_0)$. Here $\langle\cdot,\cdot\rangle$ denotes the pairing of $H_0^1(\Omega_0)$ and $H^{-1}(\Omega_0)$. We know that the transformed solution u belongs to $C^0\big([0,\infty), L^2(\Omega_0)\big)$ by the semigroup theory.

5.2 Changing the time

The main idea is to adapt the method of self-similar solutions used in the case of the heat equation in the whole Euclidean space by Escobedo and Kavian [8] to the present problem. We perform the self-similarity transformation in the first (longitudinal) space variable only, while keeping the other (transverse) space variables unchanged.

More precisely, we consider a unitary transformation $\tilde{U}$ on $L^2(\Omega_0)$ which associates to every solution $u \in L^2_{\text{loc}}\big((0,\infty), dt; L^2(\Omega_0, dx)\big)$ of (5.1) a self-similar solution $\tilde{u} := \tilde{U}u$ in a new s-time weighted space $L^2_{\text{loc}}\big((0,\infty), e^s ds; L^2(\Omega_0, dy)\big)$ via (1.12). The inverse change of variables is given by

$$u(x_1,x_2,x_3,t) = (t+1)^{-1/4}\,\tilde{u}\big((t+1)^{-1/2}x_1, x_2, x_3, \log(t+1)\big)\,.$$

When evolution is posed in that context, $y = (y_1,y_2,y_3)$ plays the role of space variable and s is the new time. One can check that, in the new variables, the evolution is governed by (1.13).

More precisely, the weak formulation (5.1) transfers to

$$\big\langle \tilde{v}, \tilde{u}'(s) - \tfrac{1}{2}\,y_1\,\partial_1\tilde{u}(s)\big\rangle + \tilde{Q}_s\big(\tilde{v},\tilde{u}(s)\big) - E_1\,e^s\,\big(\tilde{v},\tilde{u}(s)\big)_{L^2(\Omega_0)} = 0 \tag{5.2}$$

for each $\tilde{v} \in H_0^1(\Omega_0)$ and a.e. $s \in [0,\infty)$, with $\tilde{u}(0) = \tilde{u}_0 := \tilde{U}u_0 = u_0$. Here $\tilde{Q}_s(\cdot,\cdot)$ denotes the sesquilinear form associated with

$$\begin{aligned}
&\tilde{Q}_s[\tilde{u}] := \|\partial_1\tilde{u} - \sigma_s\,\partial_\tau\tilde{u}\|^2_{L^2(\Omega_0)} + e^s\,\|\nabla'\tilde{u}\|^2_{L^2(\Omega_0)} - \frac{1}{4}\,\|\tilde{u}\|^2_{L^2(\Omega_0)}\,,\\
&\tilde{u} \in \mathfrak{D}(\tilde{Q}_s) := H_0^1(\Omega_0)\,,
\end{aligned}$$

where σ_s has been introduced in (1.14).

Note that the operator $\tilde{H}_s$ in $L^2(\Omega_0)$ associated with the form $\tilde{Q}_s$ has s-time-dependent coefficients, which makes the problem different from the whole-space case. In particular, the twisting represented by the function (1.14) becomes more and more "localized" in a neighbourhood of the origin $y_1 = 0$ for large time s.

5.3 The natural weighted space

Since $\tilde{U}$ acts as a unitary transformation on $L^2(\Omega_0)$, it preserves the space norm of solutions of (5.1) and (5.2), *i.e.*,

$$\|u(t)\|_{L^2(\Omega_0)} = \|\tilde{u}(s)\|_{L^2(\Omega_0)}\,. \tag{5.3}$$

This means that we can analyse the asymptotic time behaviour of the former by studying the latter.

However, the natural space to study the evolution (5.2) is not $L^2(\Omega_0)$ but rather the weighted space (1.8). For $k \in \mathbb{Z}$, we define

$$\mathcal{H}_k := L^2\big(\Omega_0, K^k(y_1)\,dy_1dy_2dy_3\big)\,.$$

Hereafter we abuse the notation a bit by denoting by K, initially introduced as a function on Ω_θ in (1.8), the analogous function on $\mathbb{R}$ too. Note that $K^{-1/2}$ is the first eigenfunction of the harmonic-oscillator Hamiltonian

$$h := -\frac{d^2}{dy_1^2} + \frac{1}{16}\, y_1^2 \qquad \text{in} \qquad L^2(\mathbb{R}) \tag{5.4}$$

(*i.e.* the Friedrichs extension of this operator initially defined on $C_0^\infty(\mathbb{R})$). The advantage of reformulating (5.2) in $\mathcal{H}_1$ instead of $\mathcal{H}_0 = L^2(\Omega_0)$ lies in the fact that then the governing elliptic operator has compact resolvent, as we shall see below (*cf* Proposition 5.3).

Let us also introduce the weighted Sobolev space

$$\mathcal{H}_k^1 := H_0^1\big(\Omega_0, K^k(y_1)\, dy_1 dy_2 dy_3\big)\,,$$

defined as the closure of $C_0^\infty(\Omega_0)$ with respect to the norm $(\|\cdot\|_{\mathcal{H}_k}^2 + \|\nabla\cdot\|_{\mathcal{H}_k}^2)^{1/2}$. Finally, we denote by $\mathcal{H}_k^{-1}$ the dual space to $\mathcal{H}_k^1$.

5.4 The evolution in the weighted space

We want to reconsider (1.13) as a parabolic problem posed in the weighted space $\mathcal{H}_1$ instead of $\mathcal{H}_0$. We begin with a formal calculation. Choosing $\tilde{v}(y) := K(y_1)v(y)$ for the test function in (5.2), where $v \in C_0^\infty(\Omega_0)$ is arbitrary, we can formally cast (5.2) into the form

$$\big\langle v, \tilde{u}'(s)\big\rangle + a_s\big(v, \tilde{u}(s)\big) = 0\,. \tag{5.5}$$

Here $\langle\cdot,\cdot\rangle$ denotes the pairing of $\mathcal{H}_1^1$ and $\mathcal{H}_1^{-1}$, and

$$\begin{aligned} a_s(v,\tilde{u}) := {} & \big(\partial_1 v - \sigma_s\, \partial_\tau v, \partial_1 \tilde{u} - \sigma_s\, \partial_\tau \tilde{u}\big)_{\mathcal{H}_1} + e^s\, \big(\nabla' v, \nabla' \tilde{u}\big)_{\mathcal{H}_1} \\ & - E_1\, e^s\, \big(v,\tilde{u}\big)_{\mathcal{H}_1} - \frac{1}{2}\, \big(y_1 v, \sigma_s\, \partial_\tau \tilde{u}\big)_{\mathcal{H}_1} - \frac{1}{4}\, \big(v,\tilde{u}\big)_{\mathcal{H}_1}\,. \end{aligned}$$

Note that a_s is not a symmetric form.

Of course, the formulae are meaningless in general, because the solution $\tilde{u}(s)$ and its derivative $\tilde{u}'(s)$ may not belong to $\mathcal{H}_1^1$ and $\mathcal{H}_1^{-1}$, respectively. We therefore proceed conversely by showing that (5.5) is actually well posed in $\mathcal{H}_1$ and that the solution solves (5.2) too. As for the former, we have:

Proposition 5.1. *For any $u_0 \in \mathcal{H}_1$, there exists a unique function $\tilde{u}$ such that*

$$\tilde{u} \in L^2_{\mathrm{loc}}\big((0,\infty); \mathcal{H}_1^1\big) \cap C^0\big([0,\infty); \mathcal{H}_1\big)\,, \qquad \tilde{u}' \in L^2_{\mathrm{loc}}\big((0,\infty); \mathcal{H}_1^{-1}\big)\,,$$

and it satisfies (5.5) *for each $v \in \mathcal{H}_1^1$ and a.e. $s \in [0,\infty)$, and $\tilde{u}(0) = u_0$.*

Proof. First of all, let us show that a_s is well-defined as a sesquilinear form with domain $\mathfrak{D}(a_s) := \mathcal{H}_1^1$ for any fixed $s \in [0,\infty)$. In view of the boundedness of σ_s (for every finite s) and the estimate (2.4), it only requires to check that $y_1 v \in \mathcal{H}_1$ provided $v \in \mathcal{H}_1^1$. Let $v \in C_0^\infty(\Omega_0)$. Then

$$\begin{aligned} \|y_1 v\|_{\mathcal{H}_1}^2 &= 2 \int_{\Omega_0} y_1\, |v(y)|^2\, \frac{dK(y_1)}{dy_1}\, dy \\ &= -2 \int_{\Omega_0} \Big\{ |v(y)|^2 + 2\, y_1\, \Re\big[\bar{v}(y) \partial_1 v(y)\big] \Big\}\, K(y_1)\, dy \\ &\leq 4\, \|y_1\, v\|_{\mathcal{H}_1}\, \|\partial_1 v\|_{\mathcal{H}_1}\,. \end{aligned}$$

Consequently,

$$\|y_1 v\|_{\mathcal{H}_1} \leq 4\,\|\partial_1 v\|_{\mathcal{H}_1} \leq 4\,\|v\|_{\mathcal{H}_1^1}\,. \tag{5.6}$$

By density, this inequality extends to all $v \in \mathcal{H}_1^1$. Hence, $a_s(v,u)$ is well defined for all $s \geq 0$ and $v, u \in \mathcal{H}_1^1$ (we suppress the tilde over u in the rest of the proof). Then the Proposition follows by a theorem of J. L. Lions [1, Thm. X.9] about weak solutions of parabolic equations with time-dependent coefficients. We only need to verify its hypotheses:

1. Measurability. The function $s \mapsto a_s(v,u)$ is clearly measurable on $[0,\infty)$ for all $v, u \in \mathcal{H}_1^1$, since it is in fact continuous.

2. Boundedness. Let s_0 be an arbitrary positive number. Using the boundedness of $\dot\theta$, the estimates (2.4) and (5.6), it is quite easy to show that there is a constant C, depending uniquely on s_0, $\|\dot\theta\|_{L^\infty(\mathbb{R})}$ and the geometry of ω (through a and E_1), such that

$$|a_s(v,u)| \leq C\,\|v\|_{\mathcal{H}_1^1}\,\|u\|_{\mathcal{H}_1^1} \tag{5.7}$$

for all $s \in [0,s_0]$ and $v, u \in \mathcal{H}_1^1$.

3. Coercivity. Recall that a_s is not symmetric and that we consider complex functional spaces. However, since the real and imaginary parts of the solution $\tilde u$ of (5.5) evolve independently, one may restrict to real-valued functions v and $\tilde u$ there. Alternatively, it is enough to check the coercivity of the real part of a_s. We therefore need to show that there are positive constants ϵ and C such that the inequality

$$\Re\{a_s[v]\} \geq \epsilon\,\|v\|_{\mathcal{H}_1^1}^2 - C\,\|v\|_{\mathcal{H}_1}^2 \tag{5.8}$$

holds for all $v \in \mathcal{H}_1^1$ and $s \in [0,s_0]$, where $a_s[v] := a_s(v,v)$. We have

$$\begin{aligned}\Re\{a_s[v]\} = \|\partial_1 v - \sigma_s\,\partial_\tau v\|_{\mathcal{H}_1}^2 + e^s\,\|\nabla' v\|_{\mathcal{H}_1}^2 - E_1\,e^s\,\|v\|_{\mathcal{H}_1}^2 - \frac{1}{4}\,\|v\|_{\mathcal{H}_1}^2 \\ - \frac{1}{2}\,\Re\,(y_1\,v, \sigma_s\,\partial_\tau v)_{\mathcal{H}_1}\end{aligned} \tag{5.9}$$

for all $v \in \mathcal{H}_1^1$. For every $v \in C_0^\infty(\Omega_0)$, an integration by parts shows that:

$$\Re\,(y_1\,v, \sigma_s\,\partial_\tau v)_{\mathcal{H}_1} = 0\,; \tag{5.10}$$

by density, this result extends to all $v \in \mathcal{H}_1^1$. Hence, the mixed term in (5.9) vanishes. We continue with estimating the first term on the right hand side of (5.9):

$$\begin{aligned}\|\partial_1 v - \sigma_s\,\partial_\tau v\|_{\mathcal{H}_1}^2 &\geq \epsilon\,\|\partial_1 v\|_{\mathcal{H}_1}^2 - \frac{\epsilon}{1-\epsilon}\,\|\sigma_s\,\partial_\tau v\|_{\mathcal{H}_1}^2 \\ &\geq \epsilon\,\|\partial_1 v\|_{\mathcal{H}_1}^2 - \frac{\epsilon}{1-\epsilon}\,e^s\,\|\dot\theta\|_{L^\infty(\mathbb{R})}\,a^2\,\|\nabla' v\|_{\mathcal{H}_1}^2\end{aligned}$$

valid for every $\epsilon \in (0,1)$ and $v \in \mathcal{H}_1^1$. Here the second inequality follows from the definition of σ_s in (1.14) and the estimate (2.4). Using (3.1) with help of Fubini's theorem, we therefore have

$$\begin{aligned}&\|\partial_1 v - \sigma_s\,\partial_\tau v\|_{\mathcal{H}_1}^2 + (1-\epsilon)\,e^s\,\|\nabla' v\|_{\mathcal{H}_1}^2 \\ &\qquad\geq \epsilon\,\|\partial_1 v\|_{\mathcal{H}_1}^2 + E_1\,e^s\left(1 - \epsilon - \frac{\epsilon}{1-\epsilon}\,\|\dot\theta\|_{L^\infty(\mathbb{R})}\,a^2\right)\|v\|_{\mathcal{H}_1}^2\end{aligned}$$

provided that ϵ is sufficiently small (so that the expression in the round brackets is positive). Putting this inequality into (5.9), recalling (5.10) and using the trivial bounds $1 \le e^s \le e^{s_0}$ for $s \in [0, s_0]$, we conclude with

$$\Re\{a_s[v]\} \ge \epsilon \, \|\nabla v\|_{\mathcal{H}_1}^2 - \left[E_1 \, e^{s_0} \left(\epsilon + \frac{\epsilon}{1-\epsilon} \, \|\dot{\theta}\|_{L^\infty(\mathbb{R})} \, a^2 \right) + \frac{1}{4} \right] \|v\|_{\mathcal{H}_1}^2 \,,$$

valid for all sufficiently small ϵ and all real-valued $v \in \mathcal{H}_1^1$. It is clear that the last inequality can be cast into the form (5.8), with a constant ϵ depending on a and $\|\dot{\theta}\|_{L^\infty(\mathbb{R})}$, and a constant C depending on s_0, $\|\dot{\theta}\|_{L^\infty(\mathbb{R})}$ and the geometry of ω (through a and E_1).

Now it follows from [1, Thm. X.9] that the unique solution $\tilde{u}$ of (5.5) satisfies

$$\tilde{u} \in L^2\big((0, s_0); \mathcal{H}_1^1\big) \cap C^0\big([0, s_0]; \mathcal{H}_1\big) \,, \qquad \tilde{u}' \in L^2\big((0, s_0); \mathcal{H}_1^{-1}\big) \,.$$

Since s_0 is an arbitrary positive number here, we actually get a global continuous solution in the sense that $\tilde{u} \in C^0\big([0, \infty); \mathcal{H}_1\big)$. □

Remark 5.1. As a consequence of (5.7), (5.8) and the Lax-Milgram theorem, it follows that the form a_s is closed on its domain $\mathcal{H}_1^1$.

Now we are in a position to prove a partial equivalence of evolutions (5.2) and (5.5).

Proposition 5.2. *Let $u_0 \in \mathcal{H}_1$. Let $\tilde{u}$ be the unique solution to* (5.5) *for each $v \in \mathcal{H}_1^1$ and a.e. $s \in [0, \infty)$, subject to the initial condition $\tilde{u}(0) = u_0$, that is specified in Proposition 5.1. Then $\tilde{u}$ is also the unique solution to* (5.2) *for each $\tilde{v} \in \mathcal{H}_0^1$ and a.e. $s \in [0, \infty)$, subject to the same initial condition.*

Proof. Choosing $v(y) := K(y_1)^{-1} \, \tilde{v}(y)$ for the test function in (5.5), where $\tilde{v} \in C_0^\infty(\Omega_0)$ is arbitrary, one easily checks that $\tilde{u}$ satisfies (5.2) for each $\tilde{v} \in C_0^\infty(\Omega_0)$ and a.e. $s \in [0, \infty)$. By density, this result extends to all $\tilde{v} \in \mathcal{H}_0^1$. □

5.5 Reduction to a spectral problem

As a consequence of the previous subsection, reducing the space of initial data, we can focus on the asymptotic time behaviour of the solutions to (5.5). Choosing $v := \tilde{u}(s)$ in (5.5) (and possibly combining with the conjugate version of the equation if we allow non-real initial data), we arrive at the identity

$$\frac{1}{2} \frac{d}{ds} \|\tilde{u}(s)\|_{\mathcal{H}_1}^2 = -J_s^{(1)}[\tilde{u}(s)] \,, \tag{5.11}$$

where $J_s^{(1)}[\tilde{u}] := \Re\{a_s[\tilde{u}]\}$, $\tilde{u} \in \mathfrak{D}(J_s^{(1)}) := \mathfrak{D}(a_s) = \mathcal{H}_1^1$ (independent of s). Recalling (5.9) and (5.10), we have

$$J_s^{(1)}[\tilde{u}] = \|\partial_1 \tilde{u} - \sigma_s \, \partial_\tau \tilde{u}\|_{\mathcal{H}_1}^2 + e^s \, \|\nabla' \tilde{u}\|_{\mathcal{H}_1}^2 - E_1 \, e^s \, \|\tilde{u}\|_{\mathcal{H}_1}^2 - \frac{1}{4} \, \|\tilde{u}\|_{\mathcal{H}_1}^2 \,.$$

As a consequence of (5.7), (5.8) and the Lax-Milgram theorem, we know that $J_s^{(1)}$ is closed on its domain $\mathcal{H}_1^1$. It remains to analyse the coercivity of the form $J_s^{(1)}$.

More precisely, as usual for energy estimates, we replace the right hand side of (5.11) by the spectral bound, valid for each fixed $s \in [0, \infty)$,

$$\forall \tilde{u} \in \mathcal{H}_1^1, \qquad J_s^{(1)}[\tilde{u}] \geq \mu(s) \, \|\tilde{u}\|_{\mathcal{H}_1}^2 , \tag{5.12}$$

where $\mu(s)$ denotes the lowest point in the spectrum of the self-adjoint operator $T_s^{(1)}$ in $\mathcal{H}_1$ associated with $J_s^{(1)}$. Then (5.11) together with (5.12) implies the exponential bound

$$\forall s \in [0, \infty) , \qquad \|\tilde{u}(s)\|_{\mathcal{H}_1} \leq \|\tilde{u}_0\|_{\mathcal{H}_1} \, e^{-\int_0^s \mu(r) dr} , \tag{5.13}$$

In this way, the problem is reduced to a spectral analysis of the family of operators $\{T_s^{(1)}\}_{s \geq 0}$.

5.6 Removing the weight

In order to investigate the operator $T_s^{(1)}$ in $\mathcal{H}_1$, we first map it into a unitarily equivalent operator $T_s^{(0)}$ in $\mathcal{H}_0$. This can be carried out via the unitary transform $\mathcal{U}_0 : \mathcal{H}_1 \to \mathcal{H}_0$ defined by

$$(\mathcal{U}_0 u)(y) := K^{1/2}(y_1) \, u(y) \, .$$

We define $T_s^{(0)} := \mathcal{U}_0 T_s^{(1)} \mathcal{U}_0^{-1}$, which is the self-adjoint operator associated with the quadratic form $J_s^{(0)}[v] := J_s^{(1)}[\mathcal{U}_0^{-1} v]$, $v \in \mathfrak{D}(J_s^{(0)}) := \mathcal{U}_0 \, \mathfrak{D}(J_s^{(1)})$. A straightforward calculation yields

$$J_s^{(0)}[v] = \|\partial_1 v - \sigma_s \, \partial_\tau v\|_{\mathcal{H}_0}^2 + \frac{1}{16} \, \|y_1 v\|_{\mathcal{H}_0}^2 + e^s \, \|\nabla' v\|_{\mathcal{H}_0}^2 - E_1 \, e^s \, \|v\|_{\mathcal{H}_0}^2 \, . \tag{5.14}$$

It is easy to verify that the domain of $J_s^{(0)}$ coincides with the closure of $C_0^\infty(\Omega_0)$ with respect to the norm $(\|\cdot\|_{\mathcal{H}_0}^2 + \|\nabla \cdot\|_{\mathcal{H}_0}^2 + \|y_1 \cdot\|_{\mathcal{H}_0}^2)^{1/2}$. In particular, $\mathfrak{D}(J_s^{(0)})$ is independent of s. Moreover, since this closure is compactly embedded in $\mathcal{H}_0$ (one can employ the well-known fact that (5.4) has purely discrete spectrum, which essentially uses the fact that the form domain of h is compactly embedded in $L^2(\mathbb{R})$), it follows that $T_s^{(0)}$ (and therefore $T_s^{(1)}$) is an operator with compact resolvent. In particular, we have:

Proposition 5.3. $T_s^{(1)} \simeq T_s^{(0)}$ *have purely discrete spectrum for all* $s \in [0, \infty)$.

Consequently, $\mu(s)$ is the lowest eigenvalue of $T_s^{(1)}$.

5.7 The asymptotic behaviour of the spectrum

In order to study the decay rate via (5.13), we need information about the limit of the eigenvalue $\mu(s)$ as the time s tends to infinity.

Since the function σ_s from (1.14) converges in the distributional sense to a multiple of the delta function supported at zero as $s \to \infty$, it is expectable (*cf* (5.14)) that the operator $T_s^{(0)}$ will converge, in a suitable sense, to the one-dimensional operator h from (5.4) with an extra Dirichlet boundary condition at zero. More precisely, the limiting operator, denoted by h_D, is introduced as

the self-adjoint operator in $L^2(\mathbb{R})$ whose quadratic form acts in the same way as that of h but has a smaller domain

$$\mathfrak{D}(h_D^{1/2}) := \left\{\varphi \in \mathfrak{D}(h^{1/2}) \mid \varphi(0) = 0\right\}.$$

Alternatively, the form domain $\mathfrak{D}(h_D^{1/2})$ is the closure of $C_0^\infty(\mathbb{R} \setminus \{0\})$ with respect to the norm $(\|\cdot\|^2_{L^2(\mathbb{R})} + \|\nabla\cdot\|^2_{L^2(\mathbb{R})} + \|y_1\cdot\|^2_{L^2(\mathbb{R})})^{1/2}$.

To make this limit rigorous ($T_s^{(0)}$ and h_D act in different spaces), we follow [10] and decompose the Hilbert space $\mathcal{H}_0$ into an orthogonal sum

$$\mathcal{H}_0 = \mathfrak{H}_1 \oplus \mathfrak{H}_1^\perp,$$

where the subspace $\mathfrak{H}_1$ consists of functions of the form $\psi_1(y) = \varphi(y_1)\mathcal{J}_1(y')$. Recall that $\mathcal{J}_1$ denotes the positive eigenfunction of $-\Delta_D^\omega$ corresponding to E_1, normalized to 1 in $L^2(\omega)$. Given any $\psi \in \mathcal{H}_0$, we have the decomposition $\psi = \psi_1 + \phi$ with $\psi_1 \in \mathfrak{H}_1$ as above and $\phi \in \mathfrak{H}_1^\perp$. The mapping $\pi : \varphi \mapsto \psi_1$ is an isomorphism of $L^2(\mathbb{R})$ onto $\mathfrak{H}_1$. Hence, with an abuse of notations, we may identify any operator h on $L^2(\mathbb{R})$ with the operator $\pi h \pi^{-1}$ acting on $\mathfrak{H}_1 \subset \mathcal{H}_0$.

Proposition 5.4. *Let Ω_θ be twisted with $\theta \in C^1(\mathbb{R})$. Suppose that $\dot\theta$ has compact support. Then $T_s^{(0)}$ converges to $h_D \oplus 0^\perp$ in the strong-resolvent sense as $s \to \infty$,* i.e., *for every $F \in \mathcal{H}_0$,*

$$\lim_{s\to\infty} \left\| \left(T_s^{(0)} + 1\right)^{-1} F - \left[\left(h_D + 1\right)^{-1} \oplus 0^\perp\right] F \right\|_{\mathcal{H}_0} = 0.$$

Here $0^\perp$ denotes the zero operator on the subspace $\mathfrak{H}_1^\perp \subset \mathcal{H}_0$.

Proof. For any fixed $F \in \mathcal{H}_0$ and sufficiently large positive number z, let us set $\psi_s := (T_s^{(0)} + z)^{-1} F$. In other words, ψ_s satisfies the resolvent equation

$$\forall v \in \mathfrak{D}(J_s^{(0)}), \qquad J_s^{(0)}(v, \psi_s) + z\,(v, \psi_s)_{\mathcal{H}_0} = (v, F)_{\mathcal{H}_0}. \tag{5.15}$$

In particular, choosing ψ_s for the test function v in (5.15), we have

$$\|\partial_1\psi_s - \sigma_s\,\partial_\tau\psi_s\|^2_{\mathcal{H}_0} + \frac{1}{16}\,\|y_1\psi_s\|^2_{\mathcal{H}_0} + e^s\Big(\|\nabla'\psi_s\|^2_{\mathcal{H}_0} - E_1\|\psi_s\|^2_{\mathcal{H}_0}\Big) + z\,\|\psi_s\|^2_{\mathcal{H}_0}$$
$$= (\psi_s, F)_{\mathcal{H}_0} \le \frac{1}{4}\,\|\psi_s\|^2_{\mathcal{H}_0} + \|F\|^2_{\mathcal{H}_0}. \tag{5.16}$$

Henceforth we assume that $z > 1/4$.

We employ the decomposition $\psi_s(y) = \varphi_s(y_1)\mathcal{J}_1(y_1) + \phi_s(y)$ where $\phi_s \in \mathfrak{H}_1^\perp$, *i.e.*,

$$\forall y_1 \in \mathbb{R}, \qquad \big(\mathcal{J}_1, \phi_s(y_1,\cdot)\big)_{L^2(\omega)} = 0. \tag{5.17}$$

Then, for every $\epsilon \in (0,1)$,

$$\begin{aligned}\|\nabla'\psi_s\|^2_{\mathcal{H}_0} - E_1\|\psi_s\|^2_{\mathcal{H}_0} &= \epsilon\|\nabla'\phi_s\|^2_{\mathcal{H}_0} + (1-\epsilon)\|\nabla'\phi_s\|^2_{\mathcal{H}_0} - E_1\|\phi_s\|^2_{\mathcal{H}_0} \\ &\ge \epsilon\|\nabla'\phi_s\|^2_{\mathcal{H}_0} + \big[(1-\epsilon)E_2 - E_1\big]\|\phi_s\|^2_{\mathcal{H}_0},\end{aligned}$$

where E_2 denotes the second eigenvalue of $-\Delta_D^\omega$. Since E_1 is (strictly) less then E_2, we can choose the ϵ so small that (5.16) implies

$$\|\phi_s\|^2_{\mathcal{H}_0} \le Ce^{-s} \qquad \text{and} \qquad \|\nabla'\phi_s\|^2_{\mathcal{H}_0} \le Ce^{-s}, \tag{5.18}$$

where C is a constant depending on ω and $\|F\|_{\mathcal{H}_0}$. At the same time, (5.16) yields

$$\|\varphi_s\|_{L^2(\mathbb{R})} \leq C\,, \qquad \|y_1\varphi_s\|_{L^2(\mathbb{R})} \leq C\,, \qquad \text{and} \qquad \|y_1\phi_s\|_{\mathcal{H}_0} \leq C\,, \tag{5.19}$$

where C is a constant depending on $\|F\|_{\mathcal{H}_0}$.

To get an estimate on the longitudinal derivative of ψ_s, we handle the first three terms on left hand side of (5.16) as follows. Defining a new function $u_s \in \mathcal{H}_0$ by $\psi_s(y) = e^{s/4}u_s(e^{s/2}y_1, y')$ (*cf* the self-similarity transformation (1.12)) and making the change of variables $(x_1, x') = (e^{s/2}y_1, y')$, we have

$$\begin{aligned}
J_s^{(0)}[\psi_s] &= e^s\|\partial_1 u_s - \dot\theta\,\partial_\tau u_s\|_{\mathcal{H}_0}^2 + \frac{e^{-s}}{16}\,\|x_1 u_s\|_{\mathcal{H}_0}^2 + e^s\Big(\|\nabla' u_s\|_{\mathcal{H}_0}^2 - E_1\|u_s\|_{\mathcal{H}_0}^2\Big) \\
&\geq e^s\left\{\|\partial_1 u_s - \dot\theta\,\partial_\tau u_s\|_{\mathcal{H}_0}^2 + \|\nabla' u_s\|_{\mathcal{H}_0}^2 - E_1\|u_s\|_{\mathcal{H}_0}^2\right\} \\
&\geq e^s\,c_H\,\|\rho u_s\|_{\mathcal{H}_0}^2\,, \\
&= e^s\,c_H\,\|\rho_s\psi_s\|_{\mathcal{H}_0}^2\,, \qquad \text{where} \quad \rho_s(y) := \rho(e^{s/2}y_1, y')\,.
\end{aligned} \tag{5.20}$$

In the second inequality we have employed the Hardy inequality of Theorem 3.1; the constant c_H is positive by the hypothesis. Consequently, (5.16) yields

$$\|\rho_s\psi_s\|_{\mathcal{H}_0}^2 \leq Ce^{-s}\,, \tag{5.21}$$

where C is a constant depending on $\dot\theta$, ω and $\|F\|_{\mathcal{H}_0}$. Now, proceeding as in the proof of (3.8), we get

$$\begin{aligned}
&\|\partial_1\psi_s - \sigma_s\,\partial_\tau\psi_s\|_{\mathcal{H}_0}^2 + e^s\Big(\|\nabla'\psi_s\|_{\mathcal{H}_0}^2 - E_1\|\psi_s\|_{\mathcal{H}_0}^2\Big) \\
&\qquad\qquad \geq \epsilon\,\|\partial_1\psi_s\|_{\mathcal{H}_0}^2 - \frac{\epsilon}{1-\epsilon}\,\|\dot\theta\|_{L^\infty(\mathbb{R})}^2\,a^2 E_1\,e^s\,\|\psi_s\|_{L^2(I_s\times\omega)}^2
\end{aligned}$$

for every $\epsilon < \big(1 + a^2\|\dot\theta\|_{L^\infty(\mathbb{R})}^2\big)^{-1}$, where $I_s := e^{-s/2}I \equiv \{e^{-s/2}x_1 \,|\, x_1 \in I\}$ with $I := (\inf\operatorname{supp}\dot\theta, \sup\operatorname{supp}\dot\theta)$. Since

$$\|\psi_s\|_{L^2(I_s\times\omega)} \leq C\,\|\rho_s\psi_s\|_{\mathcal{H}_0}\,, \tag{5.22}$$

where C is a constant depending exclusively on I, (5.16) together with (5.21) implies $\|\partial_1\psi_s\|_{\mathcal{H}_0}^2 \leq C$, where C is a constant depending on $\dot\theta$, ω and $\|F\|_{\mathcal{H}_0}$. Recalling (5.17), we therefore get the separate bounds

$$\|\partial_1\phi_s\|_{\mathcal{H}_0} \leq C \qquad \text{and} \qquad \|\dot\varphi_s\|_{L^2(\mathbb{R})} \leq C\,, \tag{5.23}$$

with the same constant C.

By (5.18), ϕ_s converges strongly to zero in $\mathcal{H}_0$ as $s \to \infty$. Moreover, it follows from (5.18), (5.19) and (5.23) that $\{\phi_s\}_{s\geq 0}$ is a bounded family in $\mathfrak{D}(J_s^{(0)})$. Consequently, ϕ_s converges weakly to zero in $\mathfrak{D}(J_s^{(0)})$ as $s \to \infty$.

At the same time, it follows from (5.19) and (5.23) that $\{\varphi_s\}_{s\geq 0}$ is a bounded family in $\mathfrak{D}(h^{1/2})$. Therefore it is precompact in the weak topology of $\mathfrak{D}(h^{1/2})$. Let φ_∞ be a weak limit point, *i.e.*, for an increasing sequence of positive numbers $\{s_n\}_{n\in\mathbb{N}}$ such that $s_n \to \infty$ as $n \to \infty$, $\{\varphi_{s_n}\}_{n\in\mathbb{N}}$ converges weakly to φ_∞ in $\mathfrak{D}(h^{1/2})$. Actually, we may assume that it converges strongly in $L^2(\mathbb{R})$ because $\mathfrak{D}(h^{1/2})$ is compactly embedded in $L^2(\mathbb{R})$.

Employing (5.17), (5.21) together with (5.22) gives

$$\|\varphi_s\|^2_{L^2(I_s)} \leq C e^{-s}\,, \tag{5.24}$$

where C is a constant depending on $\dot\theta$, ω and $\|F\|_{\mathcal{H}_0}$. Multiplying this inequality by $e^{s/2}$ and taking the limit $s \to \infty$, we verify that

$$\varphi_\infty(0) = 0\,. \tag{5.25}$$

(We note that $\mathfrak{D}(h^{1/2}) \subset H^1(\mathbb{R})$ and that $H^1(J)$ is compactly embedded in $C^{0,\lambda}(J)$ for every $\lambda \in (0,1/2)$ and any bounded interval $J \subset \mathbb{R}$.)

Finally, let $\varphi \in C_0^\infty(\mathbb{R}\backslash\{0\})$ be arbitrary. Taking $v(x) := \varphi(x_1)\mathcal{J}_1(x')$ as the test function in (5.15), with s being replaced by s_n, and sending n to infinity, we easily check that

$$(\dot\varphi, \dot\varphi_\infty)_{L^2(\mathbb{R})} + \frac{1}{16}\,(y_1\varphi, y_1\varphi_\infty)_{L^2(\mathbb{R})} + z\,(\varphi,\varphi_\infty)_{L^2(\mathbb{R})} = (\varphi, f)_{L^2(\mathbb{R})}\,,$$

where $f(x_1) := (\mathcal{J}_1, F(x_1,\cdot))_{L^2(\omega)}$. That is, $\varphi_\infty = (h_D+z)^{-1}f$, for *any* weak limit point of $\{\varphi_s\}_{s\geq 0}$.

Summing up, we have shown that ψ_s converges strongly to ψ_∞ in $\mathcal{H}_0$ as $s \to \infty$, where $\psi_\infty(y) := \varphi_\infty(y_1)\mathcal{J}_1(y') = \big[(h_D+z)^{-1} \oplus 0^\perp\big]F$. ☐

Remark 5.2. The crucial step in the proof is certainly the usage of the Hardy inequality in the second inequality of (5.20). Indeed, it enables one to control the mixed terms coming from the first term on the left hand side of (5.16). We would like to mention that instead of the Hardy inequality itself we could have used in (5.20) the corner-stone Lemma 3.1. This would leave to the lower bound $J_s^{(0)}[\psi_s] \geq e^s\,\lambda(\dot\theta, I)\,\|\psi_s\|^2_{L^2(I_s\times\omega)}$, which is sufficient to conclude the proof in the same way as above.

Corollary 5.1. *Let Ω_θ be twisted with $\theta \in C^1(\mathbb{R})$. Suppose that $\dot\theta$ has compact support. Then*

$$\lim_{s\to\infty} \mu(s) = 3/4\,.$$

Proof. In general, the strong-resolvent convergence of Proposition 5.4 is not enough to guarantee the convergence of spectra. However, in our case, since the spectra are purely discrete, the eigenprojections converge even in norm (*cf* [23]). In particular, $\mu(s)$ converges to the first eigenvalue of h_D. It remains to notice that the first eigenvalue of h_D coincides (in view of the symmetry) with the second eigenvalue of h which is 3/4. (For the spectrum of h, see any textbook dealing with quantum harmonic oscillator, *e.g.*, [13, Sec. 2.3].) ☐

5.8 The improved decay rate - Proof of Theorem 1.1

Now we have all the prerequisites to prove Theorem 1.1. Recall that the identity $\Gamma(\Omega_\theta) = 1/4$ for untwisted tubes is already established by Corollary 4.2. Throughout this subsection we therefore assume that Ω_θ is twisted with (1.3) and show that there is an extra decay rate.

We come back to (5.13). It follows from Corollary 5.1 that for arbitrarily small positive number ε there exists a (large) positive time s_ε such that for all $s \geq s_\varepsilon$, we have $\mu(s) \geq 3/4 - \varepsilon$. Hence, fixing $\varepsilon > 0$, for all $s \geq s_\varepsilon$, we have

$$-\int_0^s \mu(r)\,dr \leq -\int_0^{s_\varepsilon} \mu(r)\,dr - (3/4-\varepsilon)(s-s_\varepsilon) \leq (3/4-\varepsilon)s_\varepsilon - (3/4-\varepsilon)s\,,$$

where the second inequality is due to the fact that $\mu(s)$ is non-negative for all $s \geq 0$ (it is in fact greater than 1/4, *cf* Proposition 5.5). At the same time, assuming $\varepsilon \leq 3/4$, we trivially have

$$-\int_0^s \mu(r)\,dr \leq 0 \leq (3/4-\varepsilon)s_\varepsilon - (3/4-\varepsilon)s$$

also for all $s \leq s_\varepsilon$. Summing up, (5.13) implies

$$\|\tilde{u}(s)\|_{\mathcal{H}_1} \leq C_\varepsilon\, e^{-(3/4-\varepsilon)s}\, \|\tilde{u}_0\|_{\mathcal{H}_1} \tag{5.26}$$

for every $s \in [0,\infty)$, where $C_\varepsilon := e^{s_\varepsilon} \geq e^{(3/4-\varepsilon)s_\varepsilon}$. Returning to the variables in the straightened tube via $u = \tilde{U}^{-1}\tilde{u}$, using (5.3) together with the point-wise estimate $1 \leq K$, and recalling that $\tilde{u}_0 = u_0$, it follows that

$$\|u(t)\|_{\mathcal{H}_0} = \|\tilde{u}(s)\|_{\mathcal{H}_0} \leq \|\tilde{u}(s)\|_{\mathcal{H}_1} \leq C_\varepsilon\,(1+t)^{-(3/4-\varepsilon)}\,\|u_0\|_{\mathcal{H}_1}$$

for every $t \in [0,\infty)$. Finally, we recall that the weight K in $\mathcal{H}_1$ depends on the longitudinal variable only, which is therefore left invariant by the mapping $\mathcal{L}_\theta$. Consequently, we apply the unitary transform (2.2) and conclude with

$$\|S(t)\|_{L^2(\Omega_\theta,K)\to L^2(\Omega_\theta)} = \sup_{u_0\in\mathcal{H}_1\setminus\{0\}} \frac{\|u(t)\|_{\mathcal{H}_0}}{\|u_0\|_{\mathcal{H}_1}} \leq C_\varepsilon\,(1+t)^{-(3/4-\varepsilon)}$$

for every $t \in [0,\infty)$. Since ε can be made arbitrarily small, this bound implies $\Gamma(\Omega_\theta) \geq 3/4$ and concludes thus the proof of Theorem 1.1.

5.9 The improved decay rate - an alternative statement

Theorem 1.1 provides quite precise information about the extra polynomial decay of solutions u of (1.2) in a twisted tube in the sense that the decay rate $\Gamma(\Omega_\theta)$ is at least three times better than in the untwisted case. On the other hand, we have no control over the constant C_Γ in (1.9) (in principle it may blow up as $\Gamma \to \Gamma(\Omega_\theta)$). As an alternative result, we therefore present also the following theorem, where we get rid of the constant C_Γ but the prize we pay is just a qualitative knowledge about the decay rate.

Theorem 5.1. *Let $\theta \in C^1(\mathbb{R})$ satisfy* (1.3)*. We have*

$$\forall t \geq 0\,, \qquad \|S(t)\|_{L^2(\Omega_\theta,K)\to L^2(\Omega_\theta)} \,\leq\, (1+t)^{-(\gamma+1/4)}\,, \tag{5.27}$$

where γ is a non-negative constant depending on $\dot{\theta}$ and ω. Moreover, γ is positive if, and only if, Ω_θ is twisted.

In order to establish Theorem 5.1, the asymptotic result of Corollary 5.1 need to be supplied with information about values of $\mu(s)$ for finite times s.

5.9.1 Singling the dimensional decay rate out

It follows from Theorem 4.1 that there is at least a 1/4 polynomial decay rate for the solutions of the heat equations. In the setting of self-similar solutions (recall (5.13) and the relation between the initial and self-similar times t and s given by (1.12)), this will be reflected in that we actually have $\mu(s) \geq 1/4$,

regardless whether the tube is twisted or not. It is therefore natural to study rather the shifted operator $T_s^{(0)} - 1/4$. However, it is not obvious from (5.14) that such an operator is non-negative.

In order to introduce the shift explicitly into the structure of the operator, we therefore introduce another unitarily equivalent operator $T_s^{(-1)} := \mathcal{U}_{-1} T_s^{(0)} (\mathcal{U}_{-1})^{-1}$ in $\mathcal{H}_{-1}$, where the map $\mathcal{U}_{-1} : \mathcal{H}_0 \to \mathcal{H}_{-1}$ acts in the same way as $\mathcal{U}_0$:

$$(\mathcal{U}_{-1} v)(y) := K^{1/2}(y_1)\, v(y)\,.$$

$T_s^{(-1)}$ is the self-adjoint operator associated with the quadratic form $J_s^{(-1)}[w] := J_s^{(0)}[(\mathcal{U}_{-1})^{-1} w]$, $w \in \mathfrak{D}(J_s^{(-1)}) := \mathcal{U}_{-1}\, \mathfrak{D}(J_s^{(0)})$. Again, it is straightforward to check that

$$J_s^{(-1)}[w] = \|\partial_1 w - \sigma_s\, \partial_\tau w\|_{\mathcal{H}_{-1}}^2 + e^s\, \|\nabla' w\|_{\mathcal{H}_{-1}}^2 - E_1\, e^s\, \|w\|_{\mathcal{H}_{-1}}^2 + \frac{1}{4}\, \|w\|_{\mathcal{H}_{-1}}^2\,.$$

Now it readily follows from the structure of the quadratic form that the shifted operator $T_s^{(-1)} - 1/4$ is non-negative. Moreover, it is positive if, and only if, the tube is twisted.

Proposition 5.5. *If Ω_θ is twisted with $\theta \in C^1(\mathbb{R})$, then we have*

$$\forall s \in [0,\infty), \qquad \mu(s) > 1/4\,.$$

Conversely, $\mu(s) = 1/4$ for all $s \in [0,\infty)$ if Ω_θ is untwisted.

Proof. Since $J_s^{(-1)}[w] - \frac{1}{4}\, \|w\|_{\mathcal{H}_{-1}}^2 \geq 0$ for every $w \in \mathfrak{D}(J_s^{(-1)})$, we clearly have $\mu(s) \geq 1/4$, regardless whether the tube is twisted or not. By definition, if it is untwisted, then either $\sigma_s = 0$ identically in $\mathbb{R}$ for all $s \in [0,\infty)$ or $\partial_\tau \mathcal{J}_1 = 0$ identically in ω, where $\mathcal{J}_1$ is the positive eigenfunction corresponding to E_1 of the Dirichlet Laplacian in $L^2(\omega)$. Consequently, choosing $w(y) = \mathcal{J}_1(y')$ as a test function for $J_s^{(-1)}$, we also get the opposite bound $\mu(s) \leq 1/4$ in the untwisted case. To get the converse result, we can proceed exactly as in the proof of Lemma 3.1: Assuming $\mu(s) = 1/4$ in the twisted case, the variational definition of the eigenvalue $\mu(s)$ would imply

$$\|\sigma_s\|_{L^2(\mathbb{R},K^{-1})} = 0 \qquad \text{or} \qquad \|\partial_\tau \mathcal{J}_1\|_{L^2(\omega)} = 0\,,$$

a contradiction. $\square$

Now we are in a position to prove Theorem 5.1.

5.9.2 Proof of Theorem 5.1

Assume (1.3). It follows from Proposition 5.5 and Corollary 5.1 that the number

$$\gamma := \inf_{s \in [0,\infty)} \mu(s) - 1/4 \tag{5.28}$$

is positive if, and only if, Ω_θ is twisted. In any case, (5.13) implies

$$\|\tilde{u}(s)\|_{\mathcal{H}_1} \leq \|\tilde{u}_0\|_{\mathcal{H}_1}\, e^{-(\gamma+1/4)s}$$

for every $s \in [0, \infty)$. Using this estimate instead of (5.26), but following the same type of arguments as in Section 5.8 below (5.26), we get

$$\|S(t)\|_{L^2(\Omega_\theta, K) \to L^2(\Omega_\theta)} \leq (1+t)^{-(\gamma+1/4)}$$

for every $t \in [0, \infty)$. This is equivalent to (5.27) and we know that γ is positive if Ω_θ is twisted. On the other hand, in view of Proposition 4.2, estimate (5.27) cannot hold with positive γ if the tube is untwisted. This concludes the proof of Theorem 5.1.

6 Conclusions

The classical interpretation of the heat equation (1.2) is that its solution u gives the evolution of the temperature distribution of a medium in the tube cooled down to zero on the boundary. It also represents the simplest version of the stochastic Fokker-Planck equation describing the Brownian motion in Ω_θ with killing boundary conditions. Then the results of the present paper can be interpreted as that the twisting implies a faster cool-down/death of the medium/Brownian particle in the tube. Many other diffusive processes in nature are governed by (1.2).

Our proof that there is an extra decay rate for solutions of (1.2) if the tube is twisted was far from being straightforward. This is a bit surprising because the result is quite expectable from the physical interpretation, if one notices that the twist (locally) enlarges the boundary of the tube, while it (locally) keeps the volume unchanged. (By "locally" we mean that it is the case for bounded tubes, otherwise both the quantities are infinite of course.) At the same time, the Hardy inequality (1.1) did not play a direct role in the proof of Theorems 1.1 and 5.1 (although, combining any of the theorems with Theorem 3.1, we eventually know that the existence of the Hardy inequality is equivalent to the extra decay rate for the heat semigroup). It would be desirable to find a more direct proof of Theorem 1.1 based on (1.1).

We conjecture that the inequality of Theorem 1.1 can be replaced by equality, *i.e.*, $\Gamma(\Omega_\theta) = 3/4$ if the tube is twisted and (1.3) holds. The study of the quantitative dependence of the constant γ from Theorem 5.1 on properties of $\dot{\theta}$ and the geometry of ω also constitutes an interesting open problem. Note that the two quantities are related by $\gamma + 1/4 \leq \Gamma(\Omega_\theta)$.

Throughout the paper we assumed (1.3). We expect that this hypothesis can be replaced by a mere vanishing of $\dot{\theta}$ at infinity to get Theorems 1.1 and 5.1 (and also Theorem 3.1). This less restrictive assumption is known to be enough to ensure (1.4) and there exist versions of (1.1) even if (1.3) is violated (*cf* [18]). However, it is quite possible that a slower decay of $\dot{\theta}$ at infinity will make the effect of twisting stronger. In particular, can $\Gamma(\Omega_\theta)$ be strictly greater than 3/4 if the tube is twisted and $\dot{\theta}$ decays to zero very slowly at infinity?

Equally, it is not clear whether Proposition 3.3 holds if (1.3) is violated. There are some further open problems related to the Hardy inequality of Theorem 3.1. In particular, it is frustrating that the proof of the theorem does not extend to all $\dot{\theta}$ merely vanishing at infinity. In this context, it would be highly desirable to establish a more quantitative version of Lemma 3.1, *i.e.* to get a positive lower bound to $\lambda(\dot{\theta}, I)$ depending explicitly on $\dot{\theta}$, $|I|$ and ω.

On the other hand, a completely different situation will appear if one allows twisted tubes for which $\dot{\theta}$ does not vanish at infinity. Then the spectrum of $-\Delta_D^{\Omega_\theta}$ can actually start strictly above E_1 (*cf* [9] or [17, Corol. 6.6]) and an extra exponential decay rate for our semigroup $S(t)$ follows at once already in $L^2(\Omega_\theta)$. In such situations it is more natural to study the decay of the semigroup associated with $-\Delta_D^{\Omega_\theta}$ shifted by the lowest point in its spectrum. As a particularly interesting situation we mention the case of periodically twisted tubes, for which a systematic analysis based on the Floquet-Bloch decomposition could be developed in the spirit of [6, 20].

We expect that the extra decay rate will be induced also in other twisted models for which Hardy inequalities have been established recently [16, 15].

It would be also interesting to study the effect of twisting in other physical models. As one possible direction of this research, let us mention the question of the long time behaviour of the solutions to the dissipative wave equation [11, 12, 19].

Let us conclude the paper by a general conjecture. We expect that there is always an improvement of the decay rate for the heat semigroup if a Hardy inequality holds:

Conjecture. *Let Ω be an open connected subset of $\mathbb{R}^d$. Let H and H_+ be two self-adjoint operators in $L^2(\Omega)$ such that* $\inf\sigma(H) = \inf\sigma(H_+) = 0$. *Assume that there is a positive smooth function $\varrho : \Omega \to \mathbb{R}$ such that $H_+ \geq \varrho$, while $H - V$ is a negative operator for any non-negative non-trivial $V \in C_0^\infty(\Omega)$. Then there exists a positive function $K : \Omega \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{\|e^{-H_+ t}\|_{L^2(\Omega,K)\to L^2(\Omega)}}{\|e^{-Ht}\|_{L^2(\Omega,K)\to L^2(\Omega)}} = 0\,.$$

A similar conjecture can be stated for the same type of operators in different Hilbert spaces. In this paper we proved the conjecture for the special situation where $H = H_0 - E_1$ and $H_+ = H_\theta - E_1$ (transformed Dirichlet Laplacians) in $L^2(\Omega)$, with $\Omega = \Omega_0$ (unbounded tube). In general, the proof seems to be a hardly accessible problem.

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